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JHEP02(2009)031

(http://iopscience.iop.org/1126-6708/2009/02/031)

PUBLISHED BY IOP PUBLISHING FOR SISSA
RECEIVED: *November 3, 2008*
ACCEPTED: *January 20, 2009*
PUBLISHED: *February 11, 2009*

Duality covariant quantum field theory on noncommutative Minkowski space

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ABSTRACT: We prove that a scalar quantum field theory defined on noncommutative Minkowski spacetime with noncommuting momentum coordinates is covariant with respect to the UV/IR duality which exchanges coordinates and momenta. The proof is based on suitable resonance expansions of charged noncommutative scalar fields in a background electric field, which yields an effective description of the field theory in terms of a coupled complex two-matrix model. The two independent matrix degrees of freedom ensure unitarity and manifest **C T**-invariance of the field theory. The formalism describes an analytic continuation of the renormalizable Grosse-Wulkenhaar models to Minkowski signature.

KEYWORDS: Matrix Models, Non-Commutative Geometry, Renormalization Regularization and Renormalons.

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Contents

1. Introduction

The renormalization of noncommutative quantum field theories has undergone enormous progress during the last few years (see e.g. [34, 35] for an overview). The mixing of ultraviolet and infrared scales prohibits the successful application of conventional renormalization schemes, such as the Wilsonian approach [32]. Grosse and Wulkenhaar understood the appearance of UV/IR mixing in scalar $\phi^{\star 4}_{2d}$ theory as an anomaly due to a missing marginal term in the Lagrangian [22, 23]. A certain UV/IR duality symmetry of the theory under symplectic Fourier transformation of the fields [26] eliminates UV/IR mixing. In order to make their propagator covariant under this duality, they added a harmonic oscillator potential to the free Lagrangian. The analysis of Grosse and Wulkenhaar has been successfully extended to a variety of other models [14, 43, 29, 30, 19, 18, 44, 15], and it is believed that a constructive definition of these quantum field theories may be possible due to the absence of renormalons [10, 31, 13]. The UV/IR duality has been recently interpreted in

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terms of metaplectic representations of the Heisenberg group in [3], where the analog of the Grosse-Wulkenhaar model has also been defined on solvable symmetric spaces.

The duality covariant propagators in the original field theories studied in [26] govern the propagation of charged scalar particles in a constant magnetic background. Heuristically, the duality exchanges infrared and ultraviolet divergences, such that both divergences can be cut off simultaneously. This enables the standard Wilsonian renormalization procedure to be properly applied. However, thus far all models considered have been formulated in Euclidean space. In this paper we will investigate how the duality covariant scalar quantum field theories are modified in Minkowski space with maximal rank noncommutativity.

In contrast to the commutative case, the perturbative dynamics of noncommutative field theories in Minkowski signature cannot be simply obtained via a Wick rotation of their Euclidean counterparts [2, 1, 36, 27]. In non-planar graphs, the Heaviside function implementing time-ordering and the two-point function cannot be combined to yield twisted convolution products of Feynman propagators. A careful analysis treating both functions on a different footing reveals that the renormalization properties in Minkowski signature are very different than on Euclidean space [1], and it has been suggested that the UV/IR mixing problem may be far less severe or even absent in this case.

In order to analyse the UV/IR duality in Minkowski signature, we will continue the models investigated in [26, 29, 30] to Minkowski space. Thus we will consider a complex scalar field in a background *electric* field. We will establish the duality covariance of the interacting noncommutative quantum field theory. In doing so we will introduce a matrix basis for the expansion of fields, which can be considered as the Minkowskian analog of the expansion in Landau wavefunctions on noncommutative Euclidean space. The matrix basis is the key setting for application of the Wilson-Polchinski renormalization group equation in the Grosse-Wulkenhaar model. In contrast to the Euclidean case, however, the Lorentzian duality covariant field theory requires *two* coupled complex matrices in its representation as a matrix model, a necessary unitary and causal property which does not follow by a simple Wick rotation. The two-matrix model naturally ensures the stability and $\boldsymbol{C}\,\boldsymbol{T}$-invariance of the field theory. This model can thus be regarded as an analytical continuation of the Grosse-Wulkenhaar models to noncommutative Minkowski space, and is the starting point for the renormalization of noncommutative quantum field theory in Lorentzian signature.

The $1+1$-dimensional Klein-Gordon operator appearing in the free part of the duality covariant action is a special representation of the quantum *inverted* harmonic oscillator defined by the Hamiltonian

$$\hat{\boldsymbol{H}} = \frac{1}{2}\left(\hat{\boldsymbol{P}}^2 - \omega^2\,\hat{\boldsymbol{Q}}^2\right) \tag{1.1}$$

with $\omega \in \mathbb{R}$, where the position and momentum operators $\hat{\boldsymbol{Q}}$ and $\hat{\boldsymbol{P}}$ obey the canonical commutation relation $[\hat{\boldsymbol{Q}},\hat{\boldsymbol{P}}] = \mathrm{i}$. The inverted harmonic oscillator emerges if one inserts an imaginary frequency $\pm\,\mathrm{i}\,\omega$ into the usual quantum harmonic oscillator. As we will see below, we can also obtain one system from the other by a complex scaling. However, the spectral properties of these two systems are completely different. Unlike the quantum

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harmonic oscillator, which has a discrete spectrum bounded from below, the inverted oscillator exhibits a continuous spectrum which is not bounded from below. Intriguingly, even though the operator $\hat{\boldsymbol{H}}$ is selfadjoint, it possesses a second set of generalized eigenfunctions corresponding to imaginary eigenvalues. These functions occur as residues of the original eigenfunctions analytically continued to the complex energy plane. Such functions are well known in the literature and are used to describe resonant states, often called Gamow states (see e.g. [5] for a review). To uncover these states we have to close the contour of integration over the eigenfunction expansion in the upper or lower complex half-plane, and the resulting discrete expansion is analogous to the expansion in Landau wavefunctions.

From a technical standpoint, the matrix basis is derived from an application of the Gel'fand-Maurin spectral theorem and an appropriate resonance expansion of fields. This expansion requires truncation of the configuration space of the field theory to a dense subspace, which we describe in detail. Thus the integration domain for the functional integral must be truncated, which may be thought of as an ingredient of the duality covariant regularization of the quantum field theory. We work in the framework of generalized functions and Gel'fand-Shilov spaces [21], which are subalgebras of Schwartz space closed under Fourier transformation and allow for the appropriate expansions in terms of harmonic oscillator wavefunctions [25]. The Gel'fand-Shilov spaces are also closed under multiplication with the noncommutative star product [38, 9, 37], and are thus natural candidates for the configuration spaces of duality covariant noncommutative field theories. These functional analytic techniques should all prove useful for further development of the renormalization programme on noncommutative Minkowski space.

The outline of the remainder of this paper is as follows. In section 2 we give a precise formulation of the noncommutative quantum field theory in 1 + 1-dimensions and state its duality symmetries. In section 3 we develop in detail the resonance expansion of our noncommutative fields and use it to prove the duality covariance of the Lorentzian quantum field theory. In section 4 we describe both physical and analytic properties of the subspace of Schwartz space on which our resonance expansions are valid. In section 5 we describe the equivalent two-matrix model which governs the dynamics of the duality covariant quantum field theory. In section 6 we describe the generalization of our results to higher-dimensional noncommutative Minkowski space. In section 7 we summarize our findings and discuss the prospects of using our analysis in further directions. Finally, two appendices at the end of the paper contain some of the more technical aspects of our development. In appendix A we describe properties and the explicit analytic forms of the generalized eigenfunctions which are used to derive the resonance expansions. In appendix B we derive the explicit expression for the free two-point Green's function in the duality covariant quantum field theory.

2. Formulation of the duality covariant field theory

In this section we will describe the scalar field theory we shall work with and its duality symmetries. Let us begin by giving a heuristic motivation behind the duality. Consider the noncommutative field theory of a complex scalar field $\phi(\boldsymbol{x})$ in D-dimensional spacetime. The noncommutativity parameters are specified by a real constant $D \times D$ antisymmetric

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matrix $\boldsymbol{\theta}$. The infrared dynamics of the quantum field theory are mediated through the interactions of noncommutative "dipoles" [33], which are extended degrees of freedom (rigid "rods") whose lengths are proportional to their transverse momentum. For a dipole of momentum $\boldsymbol{k}$, its dipole moment is $\boldsymbol{\theta}\cdot\boldsymbol{k}$ and the position coordinate $\boldsymbol{x}$ of the scalar field is Bopp shifted to the commutative variable

$$\boldsymbol{r}=\boldsymbol{x}+\boldsymbol{\theta}\cdot\boldsymbol{k}\ . \tag{2.1}$$

The dipole degrees of freedom are created by the operators [33, 42]

$$W_{\boldsymbol{k}}[\phi]=\ \mathrm{Tr}\ \exp\big(\mathrm{i}\,|\boldsymbol{k}|\,\phi(\boldsymbol{x})\big)\ =\ \mathrm{Tr}\ \exp\big(\mathrm{i}\,|\boldsymbol{k}|\,\phi(\boldsymbol{r}-\ \mathrm{i}\,\boldsymbol{\theta}\cdot\nabla_{\boldsymbol{r}})\big)\ . \tag{2.2}$$

In the case of noncommutative gauge theory, an alternative interpretation of the infrared dynamics as a non-renormalizable gravitational sector has been given recently in [39, 20].

On the other hand, the ultraviolet dynamics are governed by the elementary quantum fields ϕ, which create pointlike quanta of momenta $\boldsymbol{k}$. The ultraviolet and infrared degrees of freedom are "dual" to one another [33]. The UV/IR mixing problem can in this way be understood as a mismatch between the dressed coordinates (2.1) and the elementary momenta $\boldsymbol{k}$. We will cure this problem by making the UV/IR "duality" symmetric via substitution of the generalized momenta

$$\boldsymbol{k}\ \longmapsto\ \boldsymbol{k}+\boldsymbol{E}\cdot\boldsymbol{x}\,, \tag{2.3}$$

where the real constant $D\times D$ antisymmetric matrix $\boldsymbol{E}$ can be interpreted as an "electromagnetic" background.

For this, consider the quantum field theory of a massive, complex scalar field $\phi(\boldsymbol{x})$ minimally coupled to a constant electromagnetic field in flat Minkowski spacetime, and in the background of an inverted harmonic oscillator potential. To simplify the presentation we will focus mainly on the case of $D=1{+}1$ dimensions, commenting later on the extension to generic spacetime dimension (see section 6). The spacetime coordinates are denoted by $\boldsymbol{x}=(t,x)=(x^\mu)$. We will denote by $(\mathrm{G}_{\mu\nu})=\mathrm{diag}(1,1)$ the flat Euclidean metric and with $(\eta_{\mu\nu})=\mathrm{diag}(1,-1)$ the flat Minkowski metric. The electric field strength tensor is denoted $\boldsymbol{E}=(E_{\mu\nu})$ and

$$\mathcal{F}[\phi](\boldsymbol{k})=\frac{1}{2\pi}\int_{\mathbb{R}^2}\mathrm{d}\boldsymbol{x}\ \mathrm{e}^{-\mathrm{i}\,\boldsymbol{k}\cdot\boldsymbol{x}}\,\phi(\boldsymbol{x})\qquad\text{with}\quad \boldsymbol{k}\cdot\boldsymbol{x}=k_\mu\,x^\mu\ :=\ \eta_{\mu\nu}\,k^\mu\,x^\nu \tag{2.4}$$

is the usual Fourier transformation of the field $\phi(\boldsymbol{x})$.

The field theory is defined by the action $S=S_0+g^2\,S_{\rm int}$ with the free part given by

$$S_0=\int_{\mathbb{R}^2}\mathrm{d}\boldsymbol{x}\ \phi^*(\boldsymbol{x})\,\big(\sigma\,\mathsf{D}^2+(1-\sigma)\,\tilde{\mathsf{D}}^2+\mu^2\big)\phi(\boldsymbol{x})\,, \tag{2.5}$$

where the parameter $\sigma\in[0,1]$, $\mu^2>0$ is the mass parameter, and $\mathsf{D}^2=\eta^{\mu\nu}\,\mathsf{D}_\mu\,\mathsf{D}_\nu$ with D_μ the generalized momentum operators defined as

$$\mathsf{D}_\mu=\frac{1}{\sqrt{2}}\,\left(-\,\mathrm{i}\,\partial_\mu+E_{\mu\nu}\,x^\nu\right)\ , \tag{2.6}$$

and $(\partial_\mu) = (\partial/\partial x^\mu) = (\partial_t, \partial_x)$. The generalized momenta obey the commutation relations

$$[\mathsf{D}_\mu, \mathsf{D}_\nu] = \mathrm{i}\, E_{\mu\nu}\,, \tag{2.7}$$

which allows us to interpret the constant electric field strength $E_{\mu\nu} = E\,\epsilon_{\mu\nu}$ as a parameter which produces noncommuting momentum space coordinates. The other kinetic operator $\tilde{\mathsf{D}}^2 = \eta^{\mu\nu}\,\tilde{\mathsf{D}}_\mu\,\tilde{\mathsf{D}}_\nu$ is specified in terms of the "dual" momenta

$$\tilde{\mathsf{D}}_\mu = \frac{1}{\sqrt{2}}\,(-\,\mathrm{i}\,\partial_\mu - E_{\mu\nu}\,x^\nu) \tag{2.8}$$

which commute with the operators D_μ and are obtained from (2.6) by the charge conjugation transformation $\boldsymbol{C} : E_{\mu\nu} \mapsto -E_{\mu\nu}$.

The interaction part consists of the two inequivalent, noncommutative quartic interactions

$$S_{\rm int} = \int_{\mathbb{R}^2} \mathrm{d}\boldsymbol{x}\ \big[\alpha\,(\phi^* \star \phi \star \phi^* \star \phi)(\boldsymbol{x}) + \beta\,(\phi^* \star \phi^* \star \phi \star \phi)(\boldsymbol{x})\big] \tag{2.9}$$

weighted by the real parameters α and β. We will use the usual Grönewold-Moyal star-product which may be defined by the twisted convolution product

$$\begin{aligned} f(\boldsymbol{x}) \star g(\boldsymbol{x}) &= \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \mathrm{d}\boldsymbol{k} \int_{\mathbb{R}^2} \mathrm{d}\boldsymbol{p}\ \mathcal{F}[f](\boldsymbol{k})\,\mathcal{F}[g](\boldsymbol{p})\ \mathrm{e}^{\,\frac{\mathrm{i}}{2}\,\theta\,\epsilon_{\mu\nu}\,k^\mu\,p^\nu}\ \ \mathrm{e}^{\,\mathrm{i}\,(\boldsymbol{k}+\boldsymbol{p})\cdot\boldsymbol{x}} \\ &= \frac{1}{(2\pi\,\theta)^2} \int_{\mathbb{R}^2} \mathrm{d}\boldsymbol{x}_1 \int_{\mathbb{R}^2} \mathrm{d}\boldsymbol{x}_2\ f(\boldsymbol{x}_1)\,g(\boldsymbol{x}_2)\ \mathrm{e}^{-\frac{2\,\mathrm{i}}{\theta}\,\epsilon_{\mu\nu}\,(\boldsymbol{x}_1-\boldsymbol{x})^\mu\,(\boldsymbol{x}_2-\boldsymbol{x})^\nu}\,. \end{aligned} \tag{2.10}$$

We assume here that $\phi \in \mathcal{S}(\mathbb{R}^2)$ is a Schwartz test function on $\mathbb{R}^2$ for simplicity. The Fourier transformation (2.4) is a topological automorphism of Schwartz space, and the twist factor $\exp(\frac{\mathrm{i}}{2}\,\theta\,\epsilon_{\mu\nu}\,k^\mu\,p^\nu)$ is a multiplier for this space. Later on we will further restrict this space to an appropriate subspace.

We are now ready to give a precise formulation of the duality in the classical field theory.

Theorem 1. *The action*

$$S = S_0 + g^2\,S_{\rm int} \ \ =:\ \ S[\phi; \boldsymbol{E}, g, \boldsymbol{\theta}] \tag{2.11}$$

defined above obeys

$$S[\phi; \boldsymbol{E}, g, \boldsymbol{\theta}] = S\big[\tilde{\phi}\,;\,\boldsymbol{E}\,,\,\tilde{g}\,,\,\tilde{\boldsymbol{\theta}}\,\big]\,, \tag{2.12}$$

where

$$\tilde{\phi}(\boldsymbol{x}) = \sqrt{\big|\det(\boldsymbol{E})\big|}\ \mathcal{F}[\phi](\boldsymbol{E}\cdot\boldsymbol{x}) \tag{2.13}$$

and $\mathcal{F}[\phi](\boldsymbol{k})$ is the Fourier transform of $\phi(\boldsymbol{x})$. The transformed coupling parameters are

$$\tilde{\boldsymbol{\theta}} = -4\boldsymbol{E}^{-1}\,\boldsymbol{\theta}^{-1}\,\boldsymbol{E}^{-1} \qquad \text{and} \qquad \tilde{g} = 2\big|\det(\boldsymbol{E}\cdot\boldsymbol{\theta})\big|^{-1/2}\,g\ . \tag{2.14}$$

Moreover, the transformation $(\phi; \boldsymbol{E}, g, \boldsymbol{\theta}) \mapsto (\tilde{\phi}; \boldsymbol{E}, \tilde{g}, \tilde{\boldsymbol{\theta}}\,)$ is a duality of the field theory, i.e. it generates a cyclic group of order two.

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At the special points $\theta = \pm\, 2/E$ the field theory is completely invariant under Fourier transformation (up to the sign of θ), and it is said to be *self-dual*. The proof of Theorem 1 is identical to that of [26, Prop. 1], which holds irrespectively of the signature of the spacetime metric. As in [26], each of the differential operators D^2 and $\tilde{\mathsf{D}}^2$ is invariant under Fourier transformation up to a rescaling. The duality covers both cases $\sigma = 0$ and $\sigma = 1$ representing charged scalar fields in a background electric field alone, analogously to the Euclidean models of [26]. Since

$$\mathsf{D}^2 + \tilde{\mathsf{D}}^2 = -\partial^\mu\, \partial_\mu - E^2\, x^\mu\, x_\mu\,, \tag{2.15}$$

the choice $\sigma = \frac{1}{2}$ corresponds to scalar fields in an inverted harmonic oscillator potential alone and is closest to the conventional field theories on noncommutative Minkowski space with no background electric field. In the Euclidean setting it is this choice which renders the standard noncommutative ϕ^4-theory renormalizable to all orders of perturbation theory by giving the free propagator the necessary decay behaviour for a multiscale slicing [22], achieved by discretization of the spectrum of the free Hamiltonian via the effective infrared regularization provided by the confining harmonic oscillator potential.

Let us now turn to the duality at the full quantum level in Minkowski spacetime. Formally, the quantum field theory defined by the classical action above is duality invariant even for Minkowski metric. It is defined by the usual perturbative, formal functional integral

$$Z[J] = \int \mathcal{D}\phi\ \mathcal{D}\phi^*\ \exp\big(\,\mathrm{i}\, S[\phi; \boldsymbol{E}, g, \boldsymbol{\theta}] + \mathrm{i}\,\langle\phi, J\rangle + \mathrm{i}\,\langle J, \phi\rangle\big) \tag{2.16}$$

where $\langle f, g\rangle := \int_{\mathbb{R}^2} \mathrm{d}\boldsymbol{x}\ f^*(\boldsymbol{x})\, g(\boldsymbol{x})$, with independent external sources $J(\boldsymbol{x})$ and $J^*(\boldsymbol{x})$. The generating functional of all connected Green's functions is given by

$$\mathcal{G}[J] = -\log \frac{Z[J]}{Z[0]} \ =:\ \mathcal{G}[J; \boldsymbol{E}, g, \boldsymbol{\theta}]\,. \tag{2.17}$$

As in [26], due to the duality covariance of the classical action S, the invariance of the functional integration measure under the transformation $\phi \mapsto \tilde{\phi}$, and the fact that $\langle\phi, J\rangle = \langle\tilde{\phi}, \tilde{J}\,\rangle$, we formally obtain the identity

$$\mathcal{G}[J; \boldsymbol{E}, g, \boldsymbol{\theta}] = \mathcal{G}\big[\tilde{J}\,;\ \boldsymbol{E}\ ,\ g\ ,\ \tilde{\boldsymbol{\theta}}\,\big]\ . \tag{2.18}$$

However, a proper treatment requires a specification of ultraviolet and infrared regularizations. As we will show in the next section, there exists a duality invariant regularization which cures all possible divergences of the quantum field theory. Assuming this is properly done we then see that the regularized quantum field theory is duality invariant.

We summarize this result as follows.

Theorem 2. *There exists a regularization which is invariant under the duality transformation given in Theorem 1. Moreover, with this regularization and for Minkowski spacetime metric, all Feynman amplitudes of the quantum field theory are convergent. The corresponding regularized generating functional* $\mathcal{G}_\Lambda$ *of all connected Green's functions, where* Λ

is a cut-off parameter defined by the regularization, is therefore well-defined. It possesses the duality symmetry

$$\mathcal{G}_\Lambda[J;\boldsymbol{E},g,\boldsymbol{\theta}] = \mathcal{G}_\Lambda\big[\tilde{J}\,;\,\boldsymbol{E}\,,\,\tilde{g}\,,\,\tilde{\boldsymbol{\theta}}\,\big] \tag{2.19}$$

where $\tilde{J}(\boldsymbol{x}) = \sqrt{|\det(\boldsymbol{E})|}\;\mathcal{F}[J](\boldsymbol{E}\cdot\boldsymbol{x})$ *and* $\mathcal{F}[J]$ *is the Fourier transform of* J.

The key ingredient for the quantum duality is the existence of a regularization of the quantum field theory which respects the duality. In [26] it was shown that in Euclidean space there exists a natural regularization for the theory. Rather than expanding the fields in plane waves, it is more natural to expand them in eigenfunctions of the Landau Hamiltonian $\mathsf{D}_{\rm E}^2 := {\rm G}^{\mu\nu}\,\mathsf{D}_\mu\,\mathsf{D}_\nu$ where ${\rm G}^{\mu\nu}$ is the Euclidean metric, which diagonalizes the free part of the action. Since the Landau wavefunctions are not eigenfunctions of the operator $\mathsf{D}^2 = \eta^{\mu\nu}\,\mathsf{D}_\mu\,\mathsf{D}_\nu$, the proof given in [26] does not directly apply in Minkowski spacetime. In the next section we will develop an analogous expansion for Minkowski signature, which will allow us to prove the theorem in a similar manner. The technical details are rather intricate in this case, and we will uncover some surprising differences from the Euclidean case.

In what follows it will be useful to employ the Weyl-Wigner correspondence of noncommutative field theory [40]. It will play a central role in our analysis of both the free action where no noncommutativity shows up and in our analysis of the noncommutative interactions. In the former case it will allow us to switch easily between different representations of our Klein-Gordon operator D^2, while on the other hand we can utilize some nice properties of this mapping to give explicit expressions for the generalized eigenfunctions of D^2 in our original representation and thus prove the duality covariance of the model. In the latter case we can use the same property to map our quantum field theory onto a matrix model.

The Weyl-Wigner correspondence provides a one-to-one correspondence between the algebra of fields on $\mathbb{R}^2$ and a ring of operators with (suitably normalized) trace ${\rm Tr}$, constructed through replacing the local coordinates x^μ of $\mathbb{R}^2$ by Hermitean operators $\hat{\boldsymbol{x}}^\mu$ obeying the Heisenberg commutation relations

$$\big[\hat{\boldsymbol{x}}^\mu\,,\,\hat{\boldsymbol{x}}^\nu\big] = {\rm i}\,\theta^{\mu\nu}\ . \tag{2.20}$$

Given a Schwartz function $f(\boldsymbol{x})$, we introduce its Weyl symbol

$$\hat{\mathcal{W}}[f] = \frac{1}{2\pi}\int_{\mathbb{R}^2} {\rm d}\boldsymbol{k}\;\mathcal{F}[f](\boldsymbol{k})\;\exp\big({\rm i}\,k_\mu\,\hat{\boldsymbol{x}}^\mu\big)\,, \tag{2.21}$$

which is a compact operator. The transformation $f(\boldsymbol{x})\mapsto\hat{\mathcal{W}}[f]$ is invertible with inverse given by [40]

$$f(\boldsymbol{x}) = \frac{1}{2\pi}\int_{\mathbb{R}^2} {\rm d}\boldsymbol{k}\;\;{\rm e}^{-{\rm i}\,\boldsymbol{k}\cdot\boldsymbol{x}}\;{\rm Tr}\big(\hat{\mathcal{W}}[f]\;\exp({\rm i}\,k_\mu\,\hat{\boldsymbol{x}}^\mu)\big)\;\;=:\;\mathsf{W}[\hat{\mathcal{W}}[f]](\boldsymbol{x})\,, \tag{2.22}$$

which is often called the Wigner distribution function of the operator $\hat{\mathcal{W}}[f]$. One has [40]

$$\hat{\mathcal{W}}[f]\,\hat{\mathcal{W}}[g] = \hat{\mathcal{W}}[f\star g] \qquad\text{and}\qquad \mathsf{W}[\hat{\boldsymbol{f}}\,]\star\mathsf{W}[\hat{\boldsymbol{g}}] = \mathsf{W}[\hat{\boldsymbol{f}}\,\hat{\boldsymbol{g}}] \tag{2.23}$$

for arbitrary Schwartz functions $f(\boldsymbol{x})$, $g(\boldsymbol{x})$ and compact operators $\hat{\boldsymbol{f}}$, $\hat{\boldsymbol{g}}$, while

$$\int_{\mathbb{R}^2} \mathrm{d}\boldsymbol{x}\ f(\boldsymbol{x}) = \mathrm{Tr}\left(\hat{\mathcal{W}}[f]\right) \qquad \text{and} \qquad \mathrm{Tr}\left(\hat{\boldsymbol{f}}\,\right) = \int_{\mathbb{R}^2} \mathrm{d}\boldsymbol{x}\ \mathsf{W}[\hat{\boldsymbol{f}}\,](\boldsymbol{x})\ . \tag{2.24}$$

3. Quantum duality on noncommutative Minkowski space

In order to prove Theorem 2 for Minkowski spacetime, we will show how to expand the boson fields in a discrete set of generalized eigenfunctions of the operator D^2. One of the most important facts needed for the proof in the Euclidean case is that the operator $\mathsf{D}^2_{\rm E}$ has a discrete spectrum. By expanding the fields in this discrete basis of eigenfunctions, the field theory can be mapped onto a matrix model and regularized by cutting off the sums appearing in the Feynman amplitudes at some finite matrix rank N. The Minkowski case is much more subtle, since the spectrum of the operator D^2 is the whole real line $\mathbb{R}$. Nevertheless, we will show that there exists an appropriate space of fields in which a discrete expansion is possible. This will enable us to apply the arguments given in the Euclidean case. In the following we will extend and generalize some results of [6] which were obtained in a different context than ours.

3.1 Mapping onto the inverted harmonic oscillator

To analyse the duality invariance of our model it is necessary to fix the self-dual point $\theta^{\mu\nu} = \theta\,\epsilon^{\mu\nu}$ with $\theta = 2/E$. (When we study the interacting field theory later on, we will assume that θ and E are independent parameters.)

Lemma 1. *There exists a classical Hamiltonian* $H(\boldsymbol{x}) = \frac{1}{2}\left(x^2 - t^2\right)$ *such that the actions of* D^2 *and* $\tilde{\mathsf{D}}^2$ *on any function* $f(\boldsymbol{x})$ *is proportional to the star product of* H *with that function as*

$$\mathsf{D}^2 f(\boldsymbol{x}) = E^2\, H(\boldsymbol{x}) \star f(\boldsymbol{x}) \qquad \textit{and} \qquad \tilde{\mathsf{D}}^2 f(\boldsymbol{x}) = E^2\, f(\boldsymbol{x}) \star H(\boldsymbol{x})\ . \tag{3.1}$$

Proof. The first equality follows from an elementary calculation

$$\begin{aligned} \frac{1}{2}\left(x^2 - t^2\right) \star f(\boldsymbol{x}) &= \frac{1}{2}\left[x^2 - t^2 - 2\,\mathrm{i}\,(\theta/2)\,(x\,\partial_t + t\,\partial_x) - (\theta/2)^2\,(\partial_t^2 - \partial_x^2)\right] f(\boldsymbol{x}) \\ &= \frac{1}{2E^2}\left(-\partial^\mu\,\partial_\mu - 2\,\mathrm{i}\,E\,\epsilon_{\mu\nu}\,x^\nu\,\partial^\mu - E^2\,x^\mu\,x_\mu\right) f(\boldsymbol{x}) = \frac{1}{E^2}\,\mathsf{D}^2 f(\boldsymbol{x})\,, \end{aligned} \tag{3.2}$$

where in the second line we have set $\theta = 2/E$. An analogous calculation establishes the second equality in (3.1). □

Instead of the operators D^2 and $\tilde{\mathsf{D}}^2$, we may thus work with the classical Hamiltonian $H(\boldsymbol{x})$, or even better with its Weyl symbol $\hat{\mathcal{W}}[H] =: \hat{\boldsymbol{H}}$. This operator is given by

$$\hat{\boldsymbol{H}} = \frac{1}{2}\left(\hat{\mathcal{W}}[x]^2 - \hat{\mathcal{W}}[t]^2\right) \ = \ \frac{1}{2}\left(\hat{\boldsymbol{p}}^2 - \hat{\boldsymbol{q}}^2\right), \tag{3.3}$$

where the operators $\hat{\boldsymbol{p}} := \hat{\mathcal{W}}[x]$ and $\hat{\boldsymbol{q}} := \hat{\mathcal{W}}[t]$ obey the commutation relation

$$\big[\hat{\boldsymbol{q}}\,,\,\hat{\boldsymbol{p}}\big] = \hat{\mathcal{W}}[t\star x - x\star t] = \mathrm{i}\,\theta = 2\,\mathrm{i}\,/E\ . \tag{3.4}$$

The operator $\hat{\boldsymbol{H}}$ is known as the inverted harmonic oscillator Hamiltonian. Its spectral properties are reviewed below.

3.2 Rigged Hilbert space and resonance expansion

Resonance states were first introduced to describe decay phenomena in nuclei. They correspond to complex energy eigenvalues of a Hamiltonian. The mathematical object in which to embed such states is a rigged Hilbert space. The extension of the usual Hilbert space to a rigged Hilbert space is also necessary to deal with continuous spectra of selfadjoint operators. The spectral theorem for Hilbert spaces, which is only valid for operators with discrete spectra, can be extended to these operators by the Gel'fand-Maurin theorem (also known as the nuclear spectral theorem). See e.g. [12] and references therein for an introduction to rigged Hilbert spaces in quantum mechanics.

A rigged Hilbert space is roughly speaking a triplet of spaces

$$\Phi\ \subset\ \mathcal{H}\ \subset\ \Phi'\ , \tag{3.5}$$

where Φ is a dense, topological vector subspace of an infinite-dimensional Hilbert space $\mathcal{H}$ and Φ' is its topological dual, i.e. the space of continuous linear functionals on Φ. The action of a functional $F\in\Phi'$ on a vector $\phi\in\Phi$ will be denoted $\langle\phi|F\rangle\in\mathbb{C}$. It is the extension of the inner product on $\mathcal{H}$ to $\Phi\times\Phi'$. If $\hat{\boldsymbol{A}}$ is a selfadjoint operator on $\mathcal{H}$, then a complex number $\lambda\in\mathbb{C}$ is called a *generalized eigenvalue* of $\hat{\boldsymbol{A}}$ if there is a nonzero functional $F_\lambda\in\Phi'$, called a *generalized eigenvector*, such that for any $\phi\in\Phi$ one has

$$\langle\phi|\hat{\boldsymbol{A}}F_\lambda\rangle\ :=\ \langle\hat{\boldsymbol{A}}\phi|F_\lambda\rangle = \lambda\,\langle\phi|F_\lambda\rangle\ . \tag{3.6}$$

In this way the operator $\hat{\boldsymbol{A}}$ can be extended to the dual space Φ', and it is possible to make sense of complex eigenvalues of selfadjoint operators.

By the Gel'fand-Maurin theorem, for every selfadjoint operator $\hat{\boldsymbol{A}}$ there exists a measure $\mathrm{d}\mu$ on the spectrum $\Sigma(\hat{\boldsymbol{A}})\subset\mathbb{R}$, which for an absolutely continuous spectrum can be chosen to be Lebesgue measure, such that for almost every $\lambda\in\Sigma(\hat{\boldsymbol{A}})$ we can find a nonzero functional $|F_\lambda\rangle\in\Phi'$ with

$$\hat{\boldsymbol{A}}|F_\lambda\rangle = \lambda\,|F_\lambda\rangle\ . \tag{3.7}$$

These generalized eigenvectors cover the spectrum and form a complete set, and thus for an arbitrary vector $|\phi\rangle\in\Phi$ provide the decomposition

$$|\phi\rangle = \sum_{\lambda_n\in\Sigma_p(\hat{\boldsymbol{A}})}\langle F_{\lambda_n}|\phi\rangle\,|F_{\lambda_n}\rangle + \int_{\Sigma_c(\hat{\boldsymbol{A}})}\mathrm{d}\lambda\ \langle F_\lambda|\phi\rangle\,|F_\lambda\rangle \tag{3.8}$$

where $\Sigma_p(\hat{\boldsymbol{A}})$ and $\Sigma_c(\hat{\boldsymbol{A}})$, with $\Sigma(\hat{\boldsymbol{A}}) = \Sigma_p(\hat{\boldsymbol{A}}) \cup \Sigma_c(\hat{\boldsymbol{A}})$, are respectively the point and continuous spectrum of $\hat{\boldsymbol{A}}$. On the domain Φ, the Gel'fand-Maurin theorem allows the spectral representation for $\hat{\boldsymbol{A}}$ given by

$$\hat{\boldsymbol{A}}\big|_{\Phi} = \sum_{\lambda_n \in \Sigma_p(\hat{\boldsymbol{A}})} \lambda_n \, |F_{\lambda_n}\rangle\langle F_{\lambda_n}| + \int_{\Sigma_c(\hat{\boldsymbol{A}})} \mathrm{d}\lambda \; \lambda \, |F_\lambda\rangle\langle F_\lambda| \; . \tag{3.9}$$

We will now investigate the spectral structure and the rigged Hilbert space of the inverted harmonic oscillator Hamiltonian $\hat{\boldsymbol{H}}$. The spectral properties of $\hat{\boldsymbol{H}}$ were analysed in [7, 8].

Our first goal is to find the eigenfunctions of $\hat{\boldsymbol{H}}$ and determine the rigged Hilbert space in which an eigenvector expansion is possible. The spectrum of $\hat{\boldsymbol{H}}$ is $\mathbb{R}$ and the rigged Hilbert space is given by

$$\mathcal{S}(\mathbb{R}) \;\subset\; L^2(\mathbb{R}) \;\subset\; \mathcal{S}'(\mathbb{R}) \,, \tag{3.10}$$

where $\mathcal{S}(\mathbb{R})$ is the Schwartz space and $\mathcal{S}'(\mathbb{R})$ is the dual space of tempered distributions. We will then show that there exists a set of generalized eigenfunctions corresponding to imaginary eigenvalues. Since these eigenvalues do not belong to the spectrum, we cannot simply apply the Gel'fand-Maurin theorem to achieve a discrete expansion on the rigged Hilbert space (3.10). Nevertheless, it is the expansion in these eigenfunctions we are interested in. We will show that they arise as residues of the original eigenfunctions corresponding to the continuous eigenvalues. Through a further restriction of the domain of the Hamiltonian $\hat{\boldsymbol{H}}$, we can apply the residue theorem to reduce the continuous eigenvector expansion to a discrete one.

Lemma 2. *The operator $\hat{\boldsymbol{H}}$ is selfadoint on $L^2(\mathbb{R})$ with spectrum $\Sigma(\hat{\boldsymbol{H}}) = \mathbb{R}$.*

The proof of Lemma 2 can be found in [7]. As mentioned above, since we are dealing with a continuous spectrum we cannot expect the eigenfunctions to live in $L^2(\mathbb{R})$. We will now choose a special representation to see what the eigenfunctions of $\hat{\boldsymbol{H}}$ look like. In order to work in a similar convention to [8], we multiply $\hat{\boldsymbol{H}}$ by $E'^{\,2} := (E/2)^2$. Denoting by $|q\rangle$ the eigenbasis of $\hat{\boldsymbol{q}}$ with eigenvalue $q \in \mathbb{R}$, we get the eigenvalue equation

$$\frac{1}{2}\big(-\partial_q^2 - E'^{\,2}\, q^2\big)\chi_\pm^{\mathcal{E}}(q) = \mathcal{E}\,\chi_\pm^{\mathcal{E}}(q) \; . \tag{3.11}$$

Since the differential operator in this equation is parity invariant, each eigenvalue $\mathcal{E}$ is two-fold degenerate as indicated through the additional index $\pm$ carried by the eigenfunctions. Substituting $z = \sqrt{2\,\mathrm{i}\,E'}\; q$ the eigenvalue equation can be rearranged to the form

$$\left(\partial_z^2 + \nu + \frac{1}{2} - \frac{z^2}{4}\right)\chi_\pm^{\mathcal{E}}(z) = 0 \,, \tag{3.12}$$

where

$$\nu = -\,\mathrm{i}\,\frac{\mathcal{E}}{E'} - \frac{1}{2} \; . \tag{3.13}$$

The differential equation (3.12) is solved by the parabolic cylinder functions $D_\nu(z)$ which are defined by

$$D_\nu(z) = \frac{1}{\Gamma(-\nu)}\ \mathrm{e}^{-\frac{1}{4}z^2} \int_0^\infty \mathrm{d}t\ \mathrm{e}^{-z\,t}\ \mathrm{e}^{-\frac{1}{2}t^2}\, t^{-\nu-1}\ . \tag{3.14}$$

In particular, every solution is a linear combination of the functions $D_\nu(z)$, $D_\nu(-z)$, $D_{-\nu-1}(\mathrm{i}\,z)$ and $D_{-\nu-1}(-\mathrm{i}\,z)$. Only two of them are linearly independent. As claimed above, the spectrum is the entire real line and is thus not bounded from below. This property is exactly what we need to construct our discrete expansion.

For our purposes we will need two different sets of normalized eigenfunctions $\chi_\pm^{\mathcal{E}}$ and $\eta_\pm^{\mathcal{E}}$, both corresponding to the eigenvalue $\mathcal{E}$. They are related to each other by $\eta_\pm^{\mathcal{E}}(q) = \chi_\pm^{\mathcal{E}}(q)^*$, and are given explicitly by [8]

$$\begin{aligned}
\chi_\pm^{\mathcal{E}}(q) &= \frac{C}{\sqrt{2\pi\,E'}}\ \mathrm{i}^{\frac{\nu}{2}+\frac{1}{4}}\,\Gamma(\nu+1)\,D_{-\nu-1}\big(\mp\sqrt{-2\,\mathrm{i}\,E'}\ q\big)\,, \\
\eta_\pm^{\mathcal{E}}(q) &= \frac{C}{\sqrt{2\pi\,E'}}\ \mathrm{i}^{\frac{\nu}{2}+\frac{1}{4}}\,\Gamma(-\nu)\,D_\nu\big(\mp\sqrt{2\,\mathrm{i}\,E'}\ q\big)
\end{aligned} \tag{3.15}$$

where $C = (E'/2\pi^2)^{1/4}$. These functions satisfy the orthonormality and completeness relations

$$\int_{\mathbb{R}} \mathrm{d}q\ \chi_\pm^{\mathcal{E}_1}(q)^*\,\chi_\pm^{\mathcal{E}_2}(q) = \delta(\mathcal{E}_1-\mathcal{E}_2) \quad \text{and} \quad \int_{\mathbb{R}} \mathrm{d}\mathcal{E}\ \chi_\pm^{\mathcal{E}}(q)^*\,\chi_\pm^{\mathcal{E}}(q') = \delta(q-q'), \tag{3.16}$$

and analogous relations for $\eta_\pm^{\mathcal{E}}$. These *generalized eigenfunctions* belong to the space of tempered distributions $\mathcal{S}'(\mathbb{R})$. Applying the Gel'fand-Maurin theorem to our inverted harmonic oscillator we get two expansions for every Schwartz function $\phi\in\mathcal{S}(\mathbb{R})$ given by

$$\phi(q) = \sum_{s=\pm}\int_{\mathbb{R}} \mathrm{d}\mathcal{E}\ \big\langle \chi_s^{\mathcal{E}}\big|\,\phi\big\rangle\ \chi_s^{\mathcal{E}}(q) \quad \text{and} \quad \phi(q) = \sum_{s=\pm}\int_{\mathbb{R}} \mathrm{d}\mathcal{E}\ \big\langle \eta_s^{\mathcal{E}}\big|\,\phi\big\rangle\ \eta_s^{\mathcal{E}}(q)\,, \tag{3.17}$$

and two spectral decompositions for $\hat{\boldsymbol{H}}$ given by

$$\hat{\boldsymbol{H}} = \sum_{s=\pm}\int_{\mathbb{R}} \mathrm{d}\mathcal{E}\ \mathcal{E}\,|\chi_s^{\mathcal{E}}\rangle\langle\chi_s^{\mathcal{E}}| \qquad \text{and} \qquad \hat{\boldsymbol{H}} = \sum_{s=\pm}\int_{\mathbb{R}} \mathrm{d}\mathcal{E}\ \mathcal{E}\,|\eta_s^{\mathcal{E}}\rangle\langle\eta_s^{\mathcal{E}}|\ . \tag{3.18}$$

As mentioned before, in addition to the eigenfunctions given above the Hamiltonian $\hat{\boldsymbol{H}}$ possesses a set of generalized eigenfunctions corresponding to a discrete set of imaginary generalized eigenvalues which do not appear in its spectrum. As shown in [8], there is a connection with the spectrum of the ordinary harmonic oscillator. By introducing the Hermitean scaling operator

$$\hat{\boldsymbol{V}}_\lambda := \exp\left(\frac{\lambda}{2}\,(\hat{\boldsymbol{p}}\,\hat{\boldsymbol{q}} + \hat{\boldsymbol{q}}\,\hat{\boldsymbol{p}})\right) \tag{3.19}$$

for $\lambda\in\mathbb{R}$, we can use Hadamard's lemma to compute

$$\hat{\boldsymbol{V}}_\lambda\,\big(\hat{\boldsymbol{p}}^2-\hat{\boldsymbol{q}}^2\big)\,\hat{\boldsymbol{V}}_\lambda^{-1} \;=\; \mathrm{e}^{2\,\mathrm{i}\,\lambda\,\theta}\,\big(\hat{\boldsymbol{p}}^2 - \mathrm{e}^{-4\,\mathrm{i}\,\lambda\,\theta}\,\hat{\boldsymbol{q}}^2\big)\ . \tag{3.20}$$

Setting $\lambda = \pm \frac{\pi}{4\theta}$, we see that the inverted harmonic oscillator Hamiltonian $\hat{\boldsymbol{H}}$ is related to the ordinary harmonic oscillator Hamiltonian $\hat{\boldsymbol{H}}_{\rm osc} = \frac{1}{2}\,(\hat{\boldsymbol{p}}^2 + \hat{\boldsymbol{q}}^2)$ through

$$\pm\,\mathrm{i}\,\hat{\boldsymbol{V}}_\pm\,\hat{\boldsymbol{H}}_{\rm osc}\,\hat{\boldsymbol{V}}_\pm^{-1} = \hat{\boldsymbol{H}} \tag{3.21}$$

with $\hat{\boldsymbol{V}}_\pm := \hat{\boldsymbol{V}}_{\mp\,\pi/4\theta}$. This enables us to construct two different sets of generalized eigenfunctions of $\hat{\boldsymbol{H}}$ by acting on the eigenfunctions of the harmonic oscillator $|m\rangle$ with the operators $\hat{\boldsymbol{V}}_\pm$. This leads to

$$\hat{\boldsymbol{H}}|f_m^\pm\rangle \;:=\; \hat{\boldsymbol{H}}\,\hat{\boldsymbol{V}}_\pm|m\rangle = \pm\,\mathrm{i}\,\hat{\boldsymbol{V}}_\pm\,\hat{\boldsymbol{H}}_{\rm osc}|m\rangle = \pm\,\mathrm{i}\,\theta\left(m+\frac{1}{2}\right)|f_m^\pm\rangle\,, \tag{3.22}$$

where $\theta\,(m+\frac{1}{2}) = (2/E)\,(m+\frac{1}{2})$, $m \in \mathbb{N}_0$ is the usual harmonic oscillator spectrum.

We can now specify the generalized eigenfunctions corresponding to the imaginary eigenvalues. Again multiplying $\hat{\boldsymbol{H}}$ and $\hat{\boldsymbol{H}}_{\rm osc}$ with $E'^{\,2} = (E/2)^2$ and working in the eigenbasis of $\hat{\boldsymbol{q}}$ we have

$$\langle q|E'^{\,2}\,\hat{\boldsymbol{H}}_{\rm osc}|n\rangle = E'\left(n+\frac{1}{2}\right)\psi_n^{\rm osc}(q)\ . \tag{3.23}$$

The orthonormal eigenfunctions of the harmonic oscillator Hamiltonian are given by

$$\psi_n^{\rm osc}(q) = N_n\ \mathrm{e}^{\,-(E'/2)\,q^2}\,H_n\big(\sqrt{E'}\ q\big)\,, \tag{3.24}$$

where $N_n = \big(\sqrt{E'}/2^n\,n!\,\sqrt{\pi}\,\big)^{1/2}$ and H_n are the usual Hermite polynomials. Applying the operators $\hat{\boldsymbol{V}}_\pm$ to these functions we get

$$f_n^\pm(q) = \langle q|\hat{\boldsymbol{V}}_{\mp\,\pi/4\theta}|n\rangle = \mathrm{e}^{\,\pm\,\frac{\mathrm{i}\pi}{8}}\ \exp\left(\pm\,\frac{\mathrm{i}\,\pi}{4}\,q\,\partial_q\right)\psi_n^{\rm osc}(q) = \mathrm{e}^{\,\pm\,\frac{\mathrm{i}\pi}{8}}\ \psi_n^{\rm osc}\big(\mathrm{e}^{\,\pm\,\frac{\mathrm{i}\pi}{4}}\,q\big), \tag{3.25}$$

and thus

$$f_n^\pm(q) \;=\; N_n^\pm\ \mathrm{e}^{\,\mp\,\mathrm{i}\,(E'/2)\,q^2}\,H_n\big(\sqrt{\pm\,\mathrm{i}\,E'}\ q\big) \tag{3.26}$$

with $N_n^\pm = (\pm\,\mathrm{i}\,)^{1/4}\,N_n$. These functions belong to the dual Schwartz space $\mathcal{S}'(\mathbb{R})$.

We now note an important property. Since $\hat{\boldsymbol{V}}_\pm^{-1} = \hat{\boldsymbol{V}}_\mp = \hat{\boldsymbol{V}}_\mp^\dagger$ and $|f_n^\pm\rangle^\dagger = \langle f_n^\pm|$ we have

$$\langle f_n^\pm|\hat{\boldsymbol{H}} = \langle n|\hat{\boldsymbol{V}}_\pm\,\hat{\boldsymbol{H}} = \langle n|\hat{\boldsymbol{V}}_\mp^{-1}\,\hat{\boldsymbol{H}} = \mp\,\mathcal{E}_n\,\langle f_n^\pm|\,, \tag{3.27}$$

with $\mathcal{E}_n := (2\,\mathrm{i}\,E)\,(n+\frac{1}{2})$. Thus $\langle f_n^\pm|$ is an eigenbra of $\hat{\boldsymbol{H}}$ corresponding to the generalized eigenvalue $\mp\,\mathcal{E}_n$. We will see that this subtle issue has some remarkable consequences and will follow us through our entire treatment. Because of this property, along with the orthonormality and completeness of the eigenstates $|n\rangle$, we have

$$\langle f_n^\pm|f_m^\mp\rangle = \delta_{nm} \qquad \text{and} \qquad \sum_{n=0}^{\infty} f_n^\pm(q)^*\,f_n^\mp(q'\,) = \delta(q-q'\,)\ . \tag{3.28}$$

To further approach our goal of a discrete expansion we will analytically continue the energy eigenfunctions $\chi_\pm^{\mathcal{E}}$ and $\eta_\pm^{\mathcal{E}}$ into the complex energy plane and investigate their analytic behaviours as functions of $\mathcal{E}$. The distributions $f_n^\pm$ will arise as residues of the functions $\chi_\pm^{\mathcal{E}}$ and $\eta_\pm^{\mathcal{E}}$. We begin with the following lemma proven in [8].

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Lemma 3. *The parabolic cylinder functions $D_\lambda(z)$ are analytic functions of $\lambda \in \mathbb{C}$.*

The analytic structure of the functions (3.15) is thus entirely governed by the gamma-functions. Since the only singularities of $\Gamma(\lambda)$ are simple poles at $\lambda = -n$, $n \in \mathbb{N}_0$ with residues

$$\mathrm{Res}_{\lambda=-n}\big(\Gamma(\lambda)\big) = \frac{(-1)^n}{n!} \,, \tag{3.29}$$

and $\mathcal{E} = \mathrm{i}\, E'\,(\nu + \frac{1}{2})$, we see that $\chi_\pm^{\mathcal{E}}$ and $\eta_\pm^{\mathcal{E}}$ have poles at $\mathcal{E} = -\mathrm{i}\, E'\,(n + \frac{1}{2})$ and $\mathcal{E} = \mathrm{i}\, E'\,(n + \frac{1}{2})$ with residues

$$\begin{aligned} \mathrm{Res}_{\mathcal{E}=-\mathrm{i}\, E'\,(n+\frac{1}{2})}\big(\chi_\pm^{\mathcal{E}}(q)\big) &= \frac{C}{\sqrt{2\pi\, E'}}\,\frac{(-1)^n}{n!}\;\mathrm{i}^{\,-\frac{n}{2}-\frac{1}{4}}\, D_n\big(\mp\sqrt{-2\,\mathrm{i}\, E'}\; q\big)\,, \\ \mathrm{Res}_{\mathcal{E}=\mathrm{i}\, E'\,(n+\frac{1}{2})}\big(\eta_\pm^{\mathcal{E}}(q)\big) &= \frac{C}{\sqrt{2\pi\, E'}}\,\frac{(-1)^n}{n!}\;\mathrm{i}^{\,\frac{n}{2}+\frac{1}{4}}\, D_n\big(\mp\sqrt{2\,\mathrm{i}\, E'}\; q\big)\,. \end{aligned} \tag{3.30}$$

Now using

$$D_n(z) = 2^{-n/2}\;\mathrm{e}^{-z^2/4}\, H_n\big(z/\sqrt{2}\big) \tag{3.31}$$

for $n \in \mathbb{N}_0$, we find

$$\mathrm{Res}_{\mathcal{E}=-\mathcal{E}_n}\big(\chi_\pm^{\mathcal{E}}(q)\big) = c_n^-\, f_n^- \qquad \text{and} \qquad \mathrm{Res}_{\mathcal{E}=\mathcal{E}_n}\big(\eta_\pm^{\mathcal{E}}(q)\big) = c_n^+\, f_n^+\,, \tag{3.32}$$

where the constants $c_n^\pm$ can be gleamed off from (3.26), (3.30) and (3.31).

We would now like to extend the integration over $\mathbb{R}$ to a closed contour integral in (3.17), and then apply the residue theorem to get a discrete expansion. However, the integral over the arc at infinity must not contribute to the contour integral. To characterize this property, we introduce two Hardy classes of functions $H_\pm^2$ which may be defined as follows [5]. Given a function $f(\mathcal{E})$ of the real variable $\mathcal{E}$ which admits an analytic continuation into the open upper complex half-plane, define the function

$$I^+(y) = \int_{\mathbb{R}} \mathrm{d}x\; \big|f(x+\mathrm{i}\, y)\big|^2 \tag{3.33}$$

of $y > 0$. Then $f(\mathcal{E})$ is in the Hardy class from above H_+^2 if and only if the integrals (3.33) are uniformly bounded, or equivalently

$$\sup_{y>0}\, I^+(y) < \infty\,. \tag{3.34}$$

The Hardy class from below H_-^2 is defined in a similar manner, by substituting $I^+(y)$ with the function $I^-(y) := I^+(-y)$.

To make sense of the contour integral we define the spaces

$$\begin{aligned} \Phi_- &= \big\{\phi \in \mathcal{S}(\mathbb{R}_q)\;\big|\;\langle\chi_\pm^{\mathcal{E}}|\phi\rangle \in \mathcal{S}(\mathbb{R}_{\mathcal{E}}) \cap H_-^2\big\}\,, \\ \Phi_+ &= \big\{\phi \in \mathcal{S}(\mathbb{R}_q)\;\big|\;\langle\eta_\pm^{\mathcal{E}}|\phi\rangle \in \mathcal{S}(\mathbb{R}_{\mathcal{E}}) \cap H_+^2\big\}\,, \end{aligned} \tag{3.35}$$

which are both dense in $L^2(\mathbb{R})$. Using the residue theorem one then proves the following result [7, 8].

Theorem 3. *For any functions $\phi^\pm \in \Phi_\pm$, one has the expansions*

$$\phi^\pm(q) = \sum_{n=0}^{\infty} \langle f_n^\mp | \phi^\pm \rangle \, f_n^\pm(q) \ . \tag{3.36}$$

With these expansions we are now almost able to complete the proof of the duality covariance of the quantum field theory in Minkowski spacetime. What remains to show is how these expansions can be applied to Schwartz functions in $\mathcal{S}(\mathbb{R}^2)$, such that each term which arises is a generalized eigenfunction of the operators D^2 and $\tilde{\mathsf{D}}^2$.

3.3 Resonance expansion of Wigner distributions

In order to achieve a discrete generalized eigenfunction expansion for functions in an appropriate dense subspace of $L^2(\mathbb{R}^2)$, we will again use the Weyl-Wigner correspondence. First of all, we have to relate the domain of D^2 and $\tilde{\mathsf{D}}^2$ to the domain of $\hat{\boldsymbol{H}}$. For this, we define the space

$$L^2(\mathbb{R}) \otimes L^2(\mathbb{R})^\vee = \Big\{ \sum_{k,l\in\mathbb{N}_0} |\psi_k\rangle\langle\varphi_l| \ \Big| \ |\psi_k\rangle \in L^2(\mathbb{R}) \, , \, \langle\varphi_l| \in L^2(\mathbb{R})^\vee \Big\} \, , \tag{3.37}$$

which contains all possible linear combinations of tensor products between functions in $L^2(\mathbb{R})$ and its dual vector space $L^2(\mathbb{R})^\vee$. This space is isomorphic to $L^2(\mathbb{R}^2)$ and we may switch between these spaces via the Weyl-Wigner correspondence. We may thus identify $L^2(\mathbb{R}^2)$ with the space of Wigner distributions $\{ \mathsf{W}[\hat{\boldsymbol{\phi}}] \mid \hat{\boldsymbol{\phi}} \in L^2(\mathbb{R}) \otimes L^2(\mathbb{R})^\vee \}$. In a similar vein, by restricting to compact operators, we may identify the Schwartz space $\mathcal{S}(\mathbb{R}^2)$ with $\mathcal{S}(\mathbb{R}) \otimes \mathcal{S}(\mathbb{R})^\vee$.

Remark 1. *The integral representation [16]*

$$\mathsf{W}[\,|\psi\rangle\langle\varphi|\,] = \frac{1}{2\pi} \int_{\mathbb{R}} \mathrm{d}k \ \ \mathrm{e}^{\,\mathrm{i}\,k\,x} \, \langle t - \theta\,k/2 | \psi\rangle \, \langle \varphi | t + \theta\,k/2 \rangle \tag{3.38}$$

can be used to define the Wigner distribution of generalized functions. In particular, it can be extended to a map on the space of tempered distributions $\mathsf{W} : \mathcal{S}'(\mathbb{R}) \otimes \mathcal{S}'(\mathbb{R})^\vee \to \mathcal{S}'(\mathbb{R}^2)$. Via (2.23), these extensions also define a continuous star product on algebras of generalized functions.

Lemma 4. *The distributions $f_{n,m}^\pm(\boldsymbol{x})$ defined by*

$$f_{n,m}^\pm \ := \ \mathsf{W}[\,|f_n^\pm\rangle\langle f_m^\mp|\,] = \mathsf{W}[\,\hat{\boldsymbol{V}}_\pm |n\rangle\langle m| \hat{\boldsymbol{V}}_\pm^{-1}\,] \tag{3.39}$$

are generalized eigenfunctions of D^2 and $\tilde{\mathsf{D}}^2$ with

$$\mathsf{D}^2 f_{n,m}^\pm(\boldsymbol{x}) = \pm\,\mathcal{E}_n \, f_{n,m}^\pm(\boldsymbol{x}) \qquad \textit{and} \qquad \tilde{\mathsf{D}}^2 f_{n,m}^\pm(\boldsymbol{x}) = \pm\,\mathcal{E}_m \, f_{n,m}^\pm(\boldsymbol{x}) \, , \tag{3.40}$$

where

$$\mathcal{E}_n = 2\,\mathrm{i}\,E \left(n + \frac{1}{2} \right) \, . \tag{3.41}$$

Proof. We use (3.1) to find

$$\mathsf{D}^2 f^{\pm}_{n,m} = E^2\,\mathsf{W}[\,\hat{\boldsymbol{H}}|f^{\pm}_n\rangle\langle f^{\mp}_m|\,] \qquad \text{and} \qquad \tilde{\mathsf{D}}^2 f^{\pm}_{n,m} = E^2\,\mathsf{W}[\,|f^{\pm}_n\rangle\langle f^{\mp}_m|\hat{\boldsymbol{H}}\,]\,, \tag{3.42}$$

and from (3.27) the generalized eigenvalue equations (3.40) follow. □

Remark 2. *One may wonder why we do not consider the more general functions* $f^{s,s'}_{n,m}$ *given by* $f^{s,s'}_{n,m} = \mathsf{W}[\hat{\boldsymbol{V}}_s|n\rangle\langle m|\hat{\boldsymbol{V}}^{-1}_{s'}]$ *with* $s, s' = \pm$*. As is shown in appendix A (Lemma 6), the distributions* $f^{+,-}_{n,m}$ *and* $f^{-,+}_{n,m}$ *vanish identically, and only the generalized eigenfunctions* $f^{\pm}_{n,m} := f^{\pm,\pm}_{n,m}$ *remain.*

The resonance expansion derived in section 3.2 above can now be applied to Wigner distributions. For brevity, we will assume that $\hat{\boldsymbol{\phi}} \in \mathcal{S}(\mathbb{R})\otimes\mathcal{S}(\mathbb{R})^\vee$ is a rank one operator $\hat{\boldsymbol{\phi}} = |\psi\rangle\langle\varphi|$, but the extension to general $\hat{\boldsymbol{\phi}}$ follows straightforwardly by linearity. Expanding $\hat{\boldsymbol{\phi}}$ in parabolic cylinder functions, we have either the expansion

$$\hat{\boldsymbol{\phi}} = \sum_{s,s'=\pm} \int_{\mathbb{R}} \mathrm{d}\mathcal{E} \int_{\mathbb{R}} \mathrm{d}\mathcal{E}'\ |\chi^{\mathcal{E}}_s\rangle\,\langle\chi^{\mathcal{E}}_s|\psi\rangle\,\langle\varphi|\chi^{\mathcal{E}'}_{s'}\rangle\,\langle\chi^{\mathcal{E}'}_{s'}| \tag{3.43}$$

or

$$\hat{\boldsymbol{\phi}} = \sum_{s,s'=\pm} \int_{\mathbb{R}} \mathrm{d}\mathcal{E} \int_{\mathbb{R}} \mathrm{d}\mathcal{E}'\ |\eta^{\mathcal{E}}_s\rangle\,\langle\eta^{\mathcal{E}}_s|\psi\rangle\,\langle\varphi|\eta^{\mathcal{E}'}_{s'}\rangle\,\langle\eta^{\mathcal{E}'}_{s'}|\ . \tag{3.44}$$

The other two possible combinations are excluded since they would lead to expansions in the functions $f^{\pm,\mp}_{n,m}(\boldsymbol{x})$ for the Wigner distributions, which vanish by Remark 2 above. Now closing the contour of integration over $\mathcal{E}$ in the lower complex half-plane and over $\mathcal{E}'$ in the upper complex half-plane in the expansion (3.43), and over $\mathcal{E}$ in the upper half-plane and over $\mathcal{E}'$ in the lower half-plane in the expansion (3.44), we find the resonance expansions

$$\hat{\boldsymbol{\phi}} = \sum_{n,m=0}^{\infty} \psi^+_n\,\varphi^{-\,*}_m\,|f^-_n\rangle\langle f^+_m| \qquad \text{and} \qquad \hat{\boldsymbol{\phi}} = \sum_{n,m=0}^{\infty} \psi^-_n\,\varphi^{+\,*}_m\,|f^+_n\rangle\langle f^-_m| \tag{3.45}$$

on $\Phi_- \otimes \Phi_+^\vee$ and $\Phi_+ \otimes \Phi_-^\vee$, respectively, where

$$\psi^{\pm}_n\ :=\ \langle f^{\pm}_n|\psi\rangle \qquad \text{and} \qquad \varphi^{\pm\,*}_m\ :=\ \langle\varphi|f^{\pm}_m\rangle\ . \tag{3.46}$$

A detailed description of the appropriate domain for both expansions is given in section 4.

Theorem 4. *For the Wigner distributions* $\phi = \mathsf{W}[\hat{\boldsymbol{\phi}}]$*, the resonance expansions correspond to*

$$\phi(\boldsymbol{x}) = \sum_{n,m=0}^{\infty} \phi^+_{m,n}\,f^-_{m,n}(\boldsymbol{x}) \qquad \text{and} \qquad \phi(\boldsymbol{x}) = \sum_{n,m=0}^{\infty} \phi^-_{m,n}\,f^+_{m,n}(\boldsymbol{x}) \tag{3.47}$$

on $\Phi_- \otimes \Phi_+^\vee$ *and* $\Phi_+ \otimes \Phi_-^\vee$*, respectively, where*

$$\phi^{\pm}_{m,n}\ :=\ \langle f^{\pm}_{m,n}|\phi\rangle = \int_{\mathbb{R}^2} \mathrm{d}\boldsymbol{x}\ f^{\pm}_{m,n}(\boldsymbol{x})^*\,\phi(\boldsymbol{x})\ . \tag{3.48}$$

Proof. On the one hand, using completeness we have

$$\phi(\boldsymbol{x}) = \sum_{n,m=0}^{\infty} \mathsf{W}\big[\,|f_m^{\mp}\rangle\,\langle f_m^{\pm}|\psi\rangle\,\langle\varphi|f_n^{\mp}\rangle\,\langle f_n^{\pm}|\,\big](\boldsymbol{x}) = \sum_{n,m=0}^{\infty} \psi_m^{\pm}\,\varphi_n^{\mp\,*}\,f_{m,n}^{\mp}(\boldsymbol{x})\ . \tag{3.49}$$

On the other hand, using (3.38) one has

$$f_{n,m}^{\pm}(\boldsymbol{x})^* = \mathsf{W}[|f_n^{\pm}\rangle\langle f_m^{\mp}|](\boldsymbol{x})^* = \frac{1}{2\pi}\int_{\mathbb{R}} \mathrm{d}k\ \mathrm{e}^{\,\mathrm{i}\,k\,x}\langle t-\theta k/2|f_m^{\mp}\rangle\langle f_n^{\pm}|t+\theta k/2\rangle = f_{m,n}^{\mp}(\boldsymbol{x}), \tag{3.50}$$

and we get

$$\begin{aligned} \int_{\mathbb{R}^2} \mathrm{d}\boldsymbol{x}\ f_{m,n}^{\pm}(\boldsymbol{x})^*\,\phi(\boldsymbol{x}) &= \int_{\mathbb{R}^2} \mathrm{d}\boldsymbol{x}\ f_{n,m}^{\mp}(\boldsymbol{x}) \star \phi(\boldsymbol{x}) \\ &= \int_{\mathbb{R}^2} \mathrm{d}\boldsymbol{x}\ \sum_{k=0}^{\infty} \mathsf{W}\big[\,|f_n^{\mp}\rangle\,\langle f_m^{\pm}|\psi\rangle\,\langle\varphi|f_k^{\mp}\rangle\,\langle f_k^{\pm}|\,\big](\boldsymbol{x}) \\ &= \int_{\mathbb{R}^2} \mathrm{d}\boldsymbol{x}\ \sum_{k=0}^{\infty} \psi_m^{\pm}\,\varphi_k^{\mp\,*}\,f_{n,k}^{\mp}(\boldsymbol{x})\ . \end{aligned} \tag{3.51}$$

Since $\int_{\mathbb{R}^2} \mathrm{d}\boldsymbol{x}\ f_{k,l}^{\pm}(\boldsymbol{x}) = \delta_{kl}$, the result now follows. □

Corollary 1. *The resonance expansions in the space of Wigner distributions are given by*

$$\mathbb{1} = \sum_{n,m=0}^{\infty} |f_{n,m}^{-}\rangle\langle f_{n,m}^{+}| \qquad \textit{and} \qquad \mathbb{1} = \sum_{n,m=0}^{\infty} |f_{n,m}^{+}\rangle\langle f_{n,m}^{-}| \tag{3.52}$$

on $\Phi_- \otimes \Phi_+^{\vee}$ *and* $\Phi_+ \otimes \Phi_-^{\vee}$*, respectively, with the notation* $f_{n,m}^{\pm}(\boldsymbol{x}) = \langle \boldsymbol{x}|f_{n,m}^{\pm}\rangle$.

3.4 Regularization

We are now ready to construct the duality invariant regularization of our quantum field theory. As shown in section 3.3 above, instead of a unique expansion as in Euclidean space [26], we now have two distinct resonance expansions (3.52) on the space of Wigner distributions. However, it is easily checked using Lemma 4 that both expansions individually lead to a free action (2.5) which is not manifestly real. We will circumvent this problem in the following way, defering a detailed technical analysis to section 4. The idea is to work on a suitable dense domain Φ wherein *both* resonance expansions are possible. Naively, this space is the intersection of the spaces $\Phi_- \otimes \Phi_+^{\vee}$ and $\Phi_+ \otimes \Phi_-^{\vee}$, but this definition is problematic due to the fact that the Hardy spaces have trivial intersection $H_+^2 \cap H_-^2 = \{0\}$ [5] (see also [7, Prop. 4]). In section 4.2 we will define Φ more precisely.

The resonance expansion on the space Φ is given by inserting

$$\mathbb{1} = \frac{1}{2}\sum_{s=\pm}\ \sum_{n,m=0}^{\infty} |f_{m,n}^{s}\rangle\langle f_{m,n}^{-s}|\ . \tag{3.53}$$

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One then readily checks the manifest reality of the action functional $S_0[\phi]$ on the field domain Φ as

$$\begin{aligned} S_0 &= \langle\phi|\,\sigma\,\mathsf{D}^2+(1-\sigma)\,\tilde{\mathsf{D}}^2+\mu^2\,|\phi\rangle \\ &= \frac{1}{2}\sum_{n,m=0}^{\infty}\Big[\big(\sigma\,\mathcal{E}_m+(1-\sigma)\,\mathcal{E}_n+\mu^2\big)\,\langle\phi|f^+_{m,n}\rangle\,\langle f^-_{m,n}|\phi\rangle \\ &\qquad\qquad +\big(-\sigma\,\mathcal{E}_m-(1-\sigma)\,\mathcal{E}_n+\mu^2\big)\,\langle\phi|f^-_{m,n}\rangle\,\langle f^+_{m,n}|\phi\rangle\Big] \qquad (3.54)\\ &= \frac{1}{2}\sum_{n,m=0}^{\infty}\Big[\big(\sigma\mathcal{E}_m+(1-\sigma)\mathcal{E}_n+\mu^2\big)\phi^+_{m,n}{}^*\phi^-_{m,n}+\big(\sigma\mathcal{E}_m+(1-\sigma)\mathcal{E}_n+\mu^2\big)^*\phi^-_{m,n}{}^*\phi^+_{m,n}\Big], \end{aligned}$$

where we have used $\mathcal{E}_n^*=-\mathcal{E}_n$. Thus both resonance expansions together are required to yield a manifestly real action.

In this basis the formal functional integration measure in (2.16) may be represented as

$$\mathcal{D}\phi\ \mathcal{D}\phi^* = \prod_{n,m=0}^{\infty}\ \prod_{s=\pm} \mathrm{d}\phi^s_{n,m}\ \mathrm{d}\phi^s_{n,m}{}^*\ , \qquad (3.55)$$

and there are two non-vanishing free propagators given by

$$C^\pm(n,m) = \big\langle\,\phi^\pm_{m,n}{}^*\ \phi^\mp_{m,n}\big\rangle = 2\,\mathrm{i}\,\big(\pm\ (\sigma\,\mathcal{E}_m+(1-\sigma)\,\mathcal{E}_n)+\mu^2\big)^{-1}\ . \qquad (3.56)$$

They also arise by representing the operator $(\sigma\,\mathsf{D}^2+(1-\sigma)\,\tilde{\mathsf{D}}^2+\mu^2)^{-1}$ in the two distinct basis sets as

$$C^\pm(n,m) = \langle f^\mp_{m,n}|2\,\mathrm{i}\,\big(\sigma\,\mathsf{D}^2+(1-\sigma)\,\tilde{\mathsf{D}}^2+\mu^2\big)^{-1}|f^\pm_{m,n}\rangle\ . \qquad (3.57)$$

Thus we have two distinct propagators which, as we will see in section 4, correspond to incoming and outgoing particle and antiparticle asymptotic states. The spacetime representation of these propagators, in the limiting case $\sigma=1$ of a background electric field alone, is derived in appendix B in terms of confluent hypergeometric functions.

Following [26], the regularization scheme we shall employ is based on the operator (2.15). Each of the operators D^2 and $\tilde{\mathsf{D}}^2$ cut off the high energy modes of one of the indices on the basis functions $f^\pm_{n,m}$. The regulated propagators in Minkowski space are thus defined by

$$\begin{aligned} C^\pm_\Lambda(n,m) &:= \langle f^\mp_{m,n}|2\,\mathrm{i}\,\big(\sigma\,\mathsf{D}^2+(1-\sigma)\,\tilde{\mathsf{D}}^2+\mu^2\big)^{-1}\,L\big(\Lambda^{-2}\,|\mathsf{D}^2+\tilde{\mathsf{D}}^2|\big)|f^\pm_{m,n}\rangle \\ &= \frac{2\,\mathrm{i}}{\pm\big(\sigma\,\mathcal{E}_m+(1-\sigma)\,\mathcal{E}_n\big)+\mu^2}\,L\left(\Lambda^{-2}\ |\mathcal{E}_n+\mathcal{E}_m|\right)\ , \end{aligned} \qquad (3.58)$$

where $\Lambda\in\mathbb{R}$ is a cut-off parameter. The cut-off function L is smooth and monotonically decreasing, with $L(y)=1$ for $y<1$ and $L(y)=0$ for $y>2$. Since the differential operator $\mathsf{D}^2+\tilde{\mathsf{D}}^2$ is invariant under Fourier transformation up to a rescaling, this regularization is duality invariant.

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In this basis each Feynman amplitude can be represented in the schematic form

$$\sum_{n_1,m_1,\dots,n_K,m_K=0}^{\infty}\ \sum_{s_1,\dots,s_K=\pm}\ \prod_{k=1}^{K} C_\Lambda^{s_k}(n_k,m_k)\,(\cdots)\ , \tag{3.59}$$

where $(\cdots)$ denotes the contributions from the noncommutative interaction vertices derived from (2.9) and combinatorial factors. Since the propagator $C_\Lambda^s(n,m)$ given by (3.58) is nonzero only if $|\mathcal{E}_n+\mathcal{E}_m| = 2|E|\,(n+m+1) < 2|\Lambda|$, which at finite Λ is true solely for a finite number of distinct values of $(n,m)\in\mathbb{N}_0^2$, every Feynman amplitude is represented by a finite sum. This completes the proof of Theorem 2 and hence establishes the quantum duality in Minkowski spacetime.

4. Configuration space

We will now clear up a few loose ends in our analysis of the previous section. By Proposition 1, the resonance expansion in generalized eigenfunctions of the operators D^2 and $\tilde{\mathsf{D}}^2$ can be obtained by just naively applying Wick rotations of the corresponding results in Euclidean space. However, the expansion in Minkowski signature involves a doubling of the effective field degrees of freedom, which does not follow by a simple Wick rotation. We will argue below that this doubling is due to a separation of time flow, wherein one expansion corresponds to motion in a given time direction while the other expansion corresponds to motion in the opposite time direction. We will also give a precise definition and rigorous, analytic description of the configuation space Φ, and show that the restriction of the functional integral to this domain may be regarded as an ingredient of the duality invariant regularization, in the sense that Φ is a dense subspace of $L^2(\mathbb{R}^2)$.

4.1 CT symmetry

We begin with a heuristic explanation for the doubling of degrees of freedom ensuing from the resonance expansion (3.53) on the configuration space Φ. We recall that the two sets of eigenfunctions in (3.15) are related by $\eta_\pm^{\mathcal{E}}(q) = \chi_\pm^{\mathcal{E}}(q)^*$, so that the two subspaces in (3.35) are related by $\Phi_+ = \Phi_-{}^\dagger$. It is for this reason that each expansion in (3.52) on its own yields a complex action, while the sum is manifestly real. On the other hand, the transformation $\chi_\pm^{\mathcal{E}}(q)\mapsto\eta_\pm^{\mathcal{E}}(q)$ is equivalent to the change $\nu+1\mapsto(\nu+1)^* = -\nu$ of the parameter (3.13). In turn, this is equivalent to reflection of the electric field $E\mapsto -E$. Now using the explicit form of the generalized eigenfunctions $f_{m,n}^\pm$ (Proposition 1), we see that the time-reversal operator $\boldsymbol{T}: t\mapsto -t$ leads to

$$\boldsymbol{T}\ :\ f_{m,n}^\pm \ \xrightarrow{t\mapsto -t}\ f_{n,m}^\pm\ . \tag{4.1}$$

On the other hand, under the charge conjugation transformation $\boldsymbol{C}: E\mapsto -E$ we get

$$\boldsymbol{C}\ :\ f_{m,n}^\pm(t,x)\ \xrightarrow{E\mapsto -E}\ f_{m,n}^\mp(-t,x) = f_{n,m}^\mp(t,x)\ . \tag{4.2}$$

Thus by applying time-reversal plus charge conjugation we get the mapping $f_{m,n}^\pm\mapsto f_{m,n}^\mp$. In particular, the spaces Φ_+ and Φ_- are in this way related via a $\boldsymbol{C}\boldsymbol{T}$-transformation, and one has $\eta_\pm^{\mathcal{E}}(q) = \boldsymbol{C}\boldsymbol{T}\chi_\pm^{\mathcal{E}}(q)$.

The domain Φ is thus the smallest domain of fields in which a $\boldsymbol{C}\,\boldsymbol{T}$-invariant resonance expansion is possible. The expansion coefficients in (3.47) are related to each other by

$$\phi^{\mp}_{n,m} = \boldsymbol{C}\,\boldsymbol{T}\phi^{\pm}_{n,m} \ := \ \langle \boldsymbol{C}\,\boldsymbol{T} f^{\pm}_{n,m}|\phi\rangle \ . \tag{4.3}$$

After fixing a time orientation, we can thus interpret C^{+} as the propagator for incoming particles and C^{-} as the propagator for outgoing antiparticles. This is consistent with the properties

$$C^{\pm}(-\boldsymbol{x};-\boldsymbol{x}'\,) = C^{\pm}(\boldsymbol{x};\boldsymbol{x}'\,) \qquad \text{and} \qquad C^{\pm}(t,x;t',x'\,)^{*} = C^{\mp}(-t,x;-t',x'\,) \tag{4.4}$$

which can be read off from (B.7). Note that, by Proposition 1, the generalized eigenfunctions $f^{\pm}_{m,n}$ have the asymptotic behaviour

$$f^{\pm}_{m,n}(t\to\pm\,\infty,x) \sim \mathrm{e}^{\,\mp\,\mathrm{i}\,E\,t^{2}} \times \ (\text{polynomial in } t) \ \ . \tag{4.5}$$

This behaviour looks somewhat like the condition for outgoing and incoming scattering states, except for the t^{2} dependence in the exponential and the polynomial factor which reflect the dipole nature of the quanta in this field theory.

4.2 Definition using Gel'fand-Shilov spaces

We will now construct a suitable configuration space of fields $\Phi\subset\mathcal{S}(\mathbb{R}^{2})$ which defines a rigged Hilbert space

$$\Phi \ \subset \ L^{2}\big(\mathbb{R}^{2}\big) \ \subset \ \Phi' \ . \tag{4.6}$$

This field domain will also define the space of matrices $\mathcal{M}$ to be integrated over in the matrix model of section 5. As we demonstrate below, the appropriate configuration space Φ can be identified with a subalgebra of one of the Gel'fand-Shilov spaces $\mathcal{S}^{\alpha}_{\alpha}(\mathbb{R}^{2})$ with $\alpha\geq\frac12$, which are subspaces of Schwartz space $\mathcal{S}(\mathbb{R}^{2})=\mathcal{S}^{\infty}_{\infty}(\mathbb{R}^{2})$. Their suitability rests on the fact that they are closed under Fourier transformation and the noncommutative star product, and their elements admit an expansion in terms of the generalized eigenfunctions that we have constructed in this paper.

We begin by reviewing the general definition of the Gel'fand-Shilov spaces $\mathcal{S}^{\alpha}_{\alpha}(\mathbb{R}^{d})$, $d\geq 1$ [21], and the properties of them that we will need. This space is the set of all smooth functions $\phi(\boldsymbol{q})$ on $\mathbb{R}^{d}$ for which there exists constants $C>0$ and $M>0$ such that

$$\big\|\boldsymbol{q}^{\boldsymbol{m}}\,\partial^{\boldsymbol{n}}_{\boldsymbol{q}}\phi\big\|_{\infty} \leq C\,M^{|\boldsymbol{n}|+|\boldsymbol{m}|}\,\boldsymbol{n}^{\alpha\,\boldsymbol{n}}\,\boldsymbol{m}^{\alpha\,\boldsymbol{m}} \tag{4.7}$$

for all $\boldsymbol{n},\boldsymbol{m}\in\mathbb{N}_{0}^{d}$, where the norm is the usual supremum norm on $L^{\infty}(\mathbb{R}^{d})$. Here we use the conventional multi-index notation where, for $\boldsymbol{n}=(n_{1},\dots,n_{d})\in\mathbb{N}_{0}^{d}$ and $\boldsymbol{q}=(q_{1},\dots,q_{d})\in\mathbb{R}^{d}$, we set $\partial^{\boldsymbol{n}}_{\boldsymbol{q}}\phi(\boldsymbol{q})=\partial^{n_{1}}_{q_{1}}\cdots\partial^{n_{d}}_{q_{d}}\phi(\boldsymbol{q})$, $|\boldsymbol{n}|=n_{1}+\cdots+n_{d}$, $\boldsymbol{n}^{\alpha\,\boldsymbol{n}}=n_{1}^{\alpha\,n_{1}}\cdots n_{d}^{\alpha\,n_{d}}$, and so on (with the convention $n_{i}^{\alpha\,n_{i}}:=1$ for $n_{i}=0$). The space $\mathcal{S}^{\alpha}_{\alpha}(\mathbb{R}^{d})$ can be realized as the inductive limit of the family of Banach spaces $\mathcal{S}^{\alpha,M}_{\alpha}(\mathbb{R}^{d})$, $M>0$ consisting of smooth functions $\phi(\boldsymbol{q})$ on $\mathbb{R}^{d}$ with finite norm

$$\|\phi\|_{\alpha,M} := \sup_{\boldsymbol{n},\boldsymbol{m}\in\mathbb{N}_{0}^{d}}\,\frac{M^{|\boldsymbol{n}|+|\boldsymbol{m}|}}{\boldsymbol{n}^{\alpha\,\boldsymbol{n}}\,\boldsymbol{m}^{\alpha\,\boldsymbol{m}}}\,\big\|\boldsymbol{q}^{\boldsymbol{m}}\,\partial^{\boldsymbol{n}}_{\boldsymbol{q}}\phi\big\|_{\infty} \ . \tag{4.8}$$

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The topology on $\mathcal{S}_\alpha^\alpha(\mathbb{R}^d)$ is then the inductive limit topology. This makes $\mathcal{S}_\alpha^\alpha(\mathbb{R}^d)$ into a Fréchet space which is a subspace of Schwartz space $\mathcal{S}(\mathbb{R}^d)$.

The Fourier transform on $\mathcal{S}_\alpha^\alpha(\mathbb{R}^d)$ is defined analogously to (2.4), and it defines a topological isomorphism. Thus the spaces $\mathcal{S}_\alpha^\alpha(\mathbb{R}^d)$ form a family of Fourier transform invariant spaces contained in the Schwartz class $\mathcal{S}(\mathbb{R}^d)$, which are closed under differentiation and multiplication by a polynomial. They are thus well-suited as configuration spaces for (free) duality covariant field theories. The Gel'fand-Shilov spaces contain quasi-analytic classes, in the sense that $\mathcal{S}_\alpha^\alpha(\mathbb{R}^d)$ for $\frac{1}{2} \leq \alpha \leq 1$ are subspaces of the space of entire functions on $\mathbb{C}^d$ restricted to $\mathbb{R}^d$. The smallest non-trivial Gel'fand-Shilov space is $\mathcal{S}_{1/2}^{1/2}(\mathbb{R}^d)$, which contains, for example, the Gaussian fields $\phi(\boldsymbol{q}) = \mathrm{e}^{-\boldsymbol{q}^2}$. The spaces $\mathcal{S}_\alpha^\alpha(\mathbb{R}^d)$ have been previously proposed as suitable test function spaces for non-local relativistic quantum field theories [4, 38].

The (strong) dual $\mathcal{S}_\alpha^\alpha(\mathbb{R}^d)'$ of the Gel'fand-Shilov class $\mathcal{S}_\alpha^\alpha(\mathbb{R}^d)$ is a space of tempered ultra-distributions of Roumieu type. It contains the space of tempered distributions $\mathcal{S}'(\mathbb{R}^d)$. The Fourier transform is extended to a continuous linear transform on $\mathcal{S}_\alpha^\alpha(\mathbb{R}^d)'$ by means of the duality formula

$$\langle\phi|\mathcal{F}[F]\rangle := \langle\mathcal{F}[\phi]|F\rangle \tag{4.9}$$

for $F \in \mathcal{S}_\alpha^\alpha(\mathbb{R}^d)'$ and $\phi \in \mathcal{S}_\alpha^\alpha(\mathbb{R}^d)$. It yields an isomorphism $\mathcal{S}_\alpha^\alpha(\mathbb{R}^d)' \to \mathcal{S}_\alpha^\alpha(\mathbb{R}^d)'$.

Let us now specialize to the one-dimensional case $d = 1$. Then the topological algebras $\mathcal{S}_\alpha^\alpha(\mathbb{R})$ have the remarkable feature that the harmonic oscillator eigenfunctions (3.24) form a basis for the expansion of fields in $\mathcal{S}_\alpha^\alpha(\mathbb{R})$ [25]. Since these eigenfunctions also form a complete orthonormal system in $L^2(\mathbb{R})$, it follows that the triplet of spaces

$$\mathcal{S}_\alpha^\alpha(\mathbb{R}) \ \subset \ L^2(\mathbb{R}) \ \subset \ \mathcal{S}_\alpha^\alpha(\mathbb{R})' \tag{4.10}$$

is a rigged Hilbert space. The corresponding expansion coefficents $\langle n|\phi\rangle$ for $n \in \mathbb{N}_0$ and $\phi \in \mathcal{S}_\alpha^\alpha(\mathbb{R})$ may be characterized as follows. The nuclear space $\mathcal{M}_\alpha^\alpha$ of sequences of ultrafast falloff is the inductive limit of the family of spaces $\mathcal{M}_\alpha^{\alpha,\kappa}$, $\kappa > 0$ consisting of complex sequences $\{a_n\}_{n\in\mathbb{N}_0}$ of finite norm

$$\big\|\{a_n\}\big\|_\kappa = \left(\ \sum_{n=0}^\infty |a_n|^2 \ \mathrm{e}^{\,2\Omega(\kappa\sqrt{n}\,)}\right)^{1/2} , \tag{4.11}$$

where we have defined the function

$$\Omega(y) := \sup_{n\in\mathbb{N}_0} \ \log\big(y^n\, n^{-\alpha\, n}\big) \tag{4.12}$$

for $y > 0$. That this space can be identified with the Gel'fand-Shilov space $\mathcal{S}_\alpha^\alpha(\mathbb{R})$ is the content of the following crucial result, proven in [25].

Theorem 5. *The mapping $\phi \mapsto a_n = \langle n|\phi\rangle$, $n \in \mathbb{N}_0$ defines a topological isomorphism on the spaces $\mathcal{S}_\alpha^\alpha(\mathbb{R}) \to \mathcal{M}_\alpha^\alpha$.*

When $a_n = \langle n|\phi\rangle$ for $\phi \in \mathcal{S}_\alpha^\alpha(\mathbb{R})$, we will denote the norm (4.11) by $\|\phi\|_\kappa$. This characterization leads to the following result governing the generalized eigenfunction expansions (3.17), which enables us to replace *both* spaces $\Phi_\pm$ in (3.35) with the Gel'fand-Shilov space $\mathcal{S}_\alpha^\alpha(\mathbb{R})$.

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Theorem 6. *For any function $\phi \in \mathcal{S}_\alpha^\alpha(\mathbb{R})$, one has:*

(a) $\lim_{\mathcal{E}\to\infty} \langle \eta_\pm^{\mathcal{E}} | \phi \rangle = 0$, *where the limit is taken over generalized eigenvalues $\mathcal{E}$ in the upper complex half-plane; and*

(b) $\lim_{\mathcal{E}\to\infty} \langle \chi_\pm^{\mathcal{E}} | \phi \rangle = 0$, *where the limit is taken over generalized eigenvalues $\mathcal{E}$ in the lower complex half-plane.*

Proof. Since $\eta_\pm^{\mathcal{E}} \in \mathcal{S}'(\mathbb{R}) \subset \mathcal{S}_\alpha^\alpha(\mathbb{R})'$ and $\phi \in \mathcal{S}_\alpha^\alpha(\mathbb{R})$, we have the Parseval equation [25]

$$\langle \eta_\pm^{\mathcal{E}} | \phi \rangle = \sum_{n=0}^{\infty} \langle \eta_\pm^{\mathcal{E}} | n \rangle \, \langle n | \phi \rangle \tag{4.13}$$

with $\langle \eta_\pm^{\mathcal{E}} | n \rangle = \int_{\mathbb{R}} \mathrm{d}q \, \eta_\pm^{\mathcal{E}}(q)^* \, \psi_n^{\mathrm{osc}}(q)$. Using the Schwarz inequality and Theorem 5, it follows that for every $\kappa > 0$ one has

$$\begin{aligned} \big| \langle \eta_\pm^{\mathcal{E}} | \phi \rangle \big| &\le \sum_{n=0}^{\infty} \big| \langle \eta_\pm^{\mathcal{E}} | n \rangle \big| \, \big| \langle n | \phi \rangle \big| \\ &\le \Big(\sum_{n=0}^{\infty} \big| \langle \eta_\pm^{\mathcal{E}} | n \rangle \big|^2 \, \mathrm{e}^{-2\Omega(\kappa\sqrt{n})} \Big)^{1/2} \Big(\sum_{n=0}^{\infty} \big| \langle n | \phi \rangle \big|^2 \, \mathrm{e}^{2\Omega(\kappa\sqrt{n})} \Big)^{1/2} \\ &= \|\phi\|_\kappa \Big(\sum_{n=0}^{\infty} \big| \langle \eta_\pm^{\mathcal{E}} | n \rangle \big|^2 \, \mathrm{e}^{-2\Omega(\kappa\sqrt{n})} \Big)^{1/2} . \end{aligned} \tag{4.14}$$

We will now substitute the explicit form of the generalized eigenfunctions $\eta_\pm^{\mathcal{E}}(q)$ from (3.15).

Using the integral representation (3.14) for the parabolic cylinder functions, it is straightforward to derive the integral identity

$$\int_{\mathbb{R}} \mathrm{d}t \; D_\nu(t) = -\frac{2\sqrt{\pi}\; 2^{\frac12(\nu+1)}}{\nu\,\Gamma\big(-\frac12\nu\big)} . \tag{4.15}$$

We will also use the estimate

$$\big\| \psi_n^{\mathrm{osc}} \big\|_\infty \le C\, n^k , \tag{4.16}$$

for some constants $C > 0$ and $k \in \mathbb{N}$ which are independent of n. For brevity, in what follows we use the same symbol C to absorb all constants independent of n and of the complex parameter ν in (3.13). We then find the bound

$$\big| \langle \eta_\pm^{\mathcal{E}} | n \rangle \big| \le C\, n^k \left| \frac{\mathrm{e}^{-\mathrm{i}\pi\nu/4}\, 2^{\nu/2}\, \Gamma(-\nu)}{\nu\,\Gamma\big(-\frac12\nu\big)} \right| . \tag{4.17}$$

Using the Stirling expansion of the gamma-functions for $|\nu| \to \infty$ and the definition (4.12), we then have

$$\begin{aligned} \big| \langle \eta_\pm^{\mathcal{E}} | n \rangle \big|^2 \, \mathrm{e}^{-2\Omega(\kappa\sqrt{n})} &\le C\, n^{2k} \big| \mathrm{e}^{\mathrm{i}\pi\nu/2}\, \nu^{-\nu-2}\, \mathrm{e}^{\nu} \big| \, \mathrm{e}^{-2\Omega(\kappa\sqrt{n})} \\ &= C \big| \mathrm{e}^{\mathrm{i}\pi\nu/2}\, \nu^{-\nu-2}\, \mathrm{e}^{\nu} \big| \frac{n^{2k}}{\Big(\sup_{m\in\mathbb{N}_0} \kappa^m\, n^{m/2}\, m^{-\alpha m} \Big)^2} \end{aligned}$$

$$\leq C \left| e^{\,i\pi\nu/2}\,\nu^{-\nu-2}\ \ e^{\nu} \right| \frac{n^{2k}\,(2k+2)^{4\alpha\,(k+1)}}{\kappa^{4(k+1)}\,n^{2k+2}}$$

$$\leq C \left| e^{\,i\pi\nu/2}\,\nu^{-\nu-2}\ \ e^{\nu} \right| \frac{1}{n^2}\,, \tag{4.18}$$

where we have chosen $\kappa \geq (2k+2)^{\alpha}$. Substituting (4.18) back into (4.14), since the series $\sum_{n\in\mathbb{N}} \frac{1}{n^2}$ converges we have finally

$$\left| \left\langle \eta^{\mathcal{E}}_{\pm} \right| \phi \right\rangle \left| \leq C\, \|\phi\|_{\kappa} \left| e^{\,i\pi\nu/4}\,\nu^{-\frac{1}{2}\nu-1}\ \ e^{\nu/2} \right| \right. . \tag{4.19}$$

The right-hand side of (4.19) vanishes in the limit $\mathrm{Re}(\nu) \to +\infty$, which proves (a). With the same techniques, an analogous bound for $|\langle \chi^{\mathcal{E}}_{\pm}|\phi\rangle|$ is obtained using the explicit form for the generalized eigenfunctions $\chi^{\mathcal{E}}_{\pm}(q)$ in (3.15), which now vanishes for $\mathrm{Re}(\nu) \to -\infty$ and establishes (b). □

Corollary 2. *For any function $\phi \in \mathcal{S}^{\alpha}_{\alpha}(\mathbb{R})$, one has the resonance expansion*

$$\phi(q) = \frac{1}{2} \sum_{s=\pm} \sum_{n=0}^{\infty} \left\langle f^{-s}_n \right| \phi \right\rangle f^{s}_n(q)\ . \tag{4.20}$$

Proof. From the estimate (4.19) and the analogous one for $|\langle \chi^{\mathcal{E}}_{\pm}|\phi\rangle|$, together with the explicit forms of the generalized eigenfunctions in (3.15), we see that the integrands of (3.17) evaluated on an arc of radius $r \to \infty$ in the upper or lower half-planes respectively vanish much faster than $r^{-1-\epsilon}$ for $\epsilon > 0$. As in the proof of [7, Thm. 2], the contributions to the contour integrals from the arcs at infinity thus vanish. □

We can now transport the resonance expansion (4.20) to the appropriate space of Wigner distributions, exactly as we did in section 3.3. The following result, whose proof may be found in [41], is helpful for this purpose.

Lemma 5. *Let $\psi, \varphi \in \mathcal{S}^{\alpha}_{\alpha}(\mathbb{R})'$. Then $\psi \in \mathcal{S}^{\alpha}_{\alpha}(\mathbb{R})$ if and only if $\mathsf{W}[\,|\psi\rangle\langle\varphi|\,] \in \mathcal{S}^{\alpha}_{\alpha}(\mathbb{R}^2)$.*

It follows from Lemma 5 that the Wigner distribution (3.50) induces a transformation

$$\mathsf{W}\ :\ \mathcal{S}^{\alpha}_{\alpha}(\mathbb{R}) \otimes \mathcal{S}^{\alpha}_{\alpha}(\mathbb{R})' \ \longrightarrow\ \mathcal{S}^{\alpha}_{\alpha}\big(\mathbb{R}^2\big) \tag{4.21}$$

for $\alpha \geq \frac{1}{2}$. In this way the space of duality covariant noncommutative scalar fields can be identified with the subspace $\Phi = \mathsf{W}\big(\mathcal{S}^{\alpha}_{\alpha}(\mathbb{R}) \otimes \mathcal{S}^{\alpha}_{\alpha}(\mathbb{R})^{\vee}\,\big)$ of the Gel'fand-Shilov space $\mathcal{S}^{\alpha}_{\alpha}(\mathbb{R}^2)$. This defines a topological algebra which is continuously closed under the star product, because of the projector property $f^{\pm}_{n,m} \star f^{\pm}_{k,l} = \delta_{mk}\, f^{\pm}_{n,l}$. This property is consistent with the result of [38] that the star product has a unique continuous extension to any Gel'fand-Shilov space $\mathcal{S}^{\alpha}_{\alpha}(\mathbb{R}^2)$. Using Theorem 5, the corresponding space of sequences $\{\phi^{\pm}_{n,m}\}_{n,m\in\mathbb{N}_0}$ can be identified with a subspace $\mathcal{M}$ of the nuclear space of sequences on $\mathbb{N}_0^2$ of ultrafast falloff [25].

5. The two-matrix model

As in the Euclidean case [30], the formalism developed in this paper enables a reformulation of the duality covariant quantum field theory as a succinct matrix model, though now with some crucial differences. Thus far we have used the Weyl-Wigner correspondence at the self-dual point $\theta = 2/E$ to find the generalized eigenbasis of the operator D^2, which depends on the absolute value of the electric field E. However, the star product appearing in the interaction term (2.9) does not depend on the electric field, but on the noncommutativity parameter θ. The generalized eigenbasis of D^2 has a very nice projector property under the star product if the noncommutativity parameter is set equal to $2/E$. For $\theta \neq 2/E$, this is no longer true in general, while the free part of the action (2.5) is still diagonal. We will now reverse the logic. We will suppose that the basis functions $f^{\pm}_{n,m}$ are defined with respect to the electric field $2/\theta \neq E$. In this case the complete action can be mapped onto a coupled complex two-matrix model.

We will fix $E \neq \pm\, 2/\theta$ generically, and assume that $E, \theta > 0$. The interaction part of our action (2.9) can be mapped onto a matrix model by noting the identities

$$\begin{aligned} \int_{\mathbb{R}^2} \mathrm{d}\boldsymbol{x}\ f^s_{n,m}(\boldsymbol{x}) &= \delta_{nm}\,, \\ f^s_{n,m} \star f^s_{k,l} &= \delta_{mk}\, f^s_{n,l}\,, \\ {f^s_{n,m}}^{\,*} &= f^{-s}_{m,n} \end{aligned} \tag{5.1}$$

for $s = \pm$. Thus the only surviving combinations for star products of four distributions $f^s_{n,m}$ are

$$f^{\pm}_{n_1,m_1} \star f^{\pm}_{n_2,m_2} \star f^{\pm}_{n_3,m_3} \star f^{\pm}_{n_4,m_4} = \delta_{n_2,m_1}\, \delta_{n_3,m_2}\, \delta_{n_4,m_3}\ f^{\pm}_{n_1,m_4}\ . \tag{5.2}$$

By using one of the two expansions $\phi(\boldsymbol{x}) = \sum_{n,m\in\mathbb{N}_0} f^s_{n,m}(\boldsymbol{x})\, \phi^{-s}_{n,m}$ for the scalar fields, we can express the interaction term $\int_{\mathbb{R}^2} \mathrm{d}\boldsymbol{x}\ (\phi^* \star \phi \star \phi^* \star \phi)(\boldsymbol{x})$ as a matrix product $\mathrm{Tr}\,\big(\phi^\dagger_{-s}\, \phi_s\, \phi^\dagger_{-s}\, \phi_s\big)$ for $s = \pm$ and $(\phi_s)_{n,m} := \phi^s_{n,m}$. The interaction $\int_{\mathbb{R}^2} \mathrm{d}\boldsymbol{x}\ (\phi^* \star \phi^* \star \phi \star \phi)(\boldsymbol{x})$ gives $\mathrm{Tr}\,\big(\phi^\dagger_{-s}\, \phi^\dagger_{-s}\, \phi_s\, \phi_s\,\big)$. On the domain Φ, the action (2.9) can thus be written as a matrix model

$$S_{\rm int} = \frac{1}{2} \sum_{s=\pm} \mathrm{Tr}\,\big(\alpha\, \phi^\dagger_{-s}\, \phi_s\, \phi^\dagger_{-s}\, \phi_s + \beta\, \phi^\dagger_{-s}\, \phi^\dagger_{-s}\, \phi_s\, \phi_s\,\big)\ . \tag{5.3}$$

The free action (2.5) for $E \neq \pm\, 2/\theta$ can also be written as a matrix product in the following way. With the help of the operators (A.9) we can write

$$\mathsf{D}^2 = \frac{1}{4\theta} \Big[(2+\theta E)^2 \Big(a^+_1 a^-_1 + \frac{\mathrm{i}}{2}\Big) + (2-\theta E)^2 \Big(a^+_2 a^-_2 + \frac{\mathrm{i}}{2}\Big) + \big(\theta^2 E^2 - 4\big)\big(a^+_1 a^+_2 + a^-_1 a^-_2\big)\Big]. \tag{5.4}$$

Note that at the two self-dual points $\theta = \pm\, 2/E$ the free action simplifies considerably, since

$$\begin{aligned} \mathsf{D}^2 &= 2E\,\Big(a^+_1\, a^-_1 + \frac{\mathrm{i}}{2}\Big) \qquad \text{for} \quad \theta = 2/E\,, \\ \mathsf{D}^2 &= 2E\,\Big(a^+_2\, a^-_2 + \frac{\mathrm{i}}{2}\Big) \qquad \text{for} \quad \theta = -2/E\ . \end{aligned} \tag{5.5}$$

The corresponding expressions for the operator $\tilde{\mathsf{D}}^2$ are obtained by interchanging $a_1^\pm \leftrightarrow a_2^\pm$ above. Using (A.5) we find

$$a_1^\pm \, a_2^\pm \, f_{n,m}^s \;=\; \mathrm{i}\, s \sqrt{n+\frac12 \pm \frac s2}\;\sqrt{m+\frac12 \pm \frac s2}\; f_{n\pm s,m\pm s}^s \,, \tag{5.6}$$

with the abbreviated notation $n \pm s := n \pm 1$ for $s = +$ and $n \pm s := n \mp 1$ for $s = -$. The free action of our model can thus be expressed as

$$\begin{aligned} S_0 = \frac{1}{8\theta} \sum_{s=\pm} \mathrm{Tr}\, \Big(& 4\theta\, \mu^2\, \phi_s^\dagger\, \phi_{-s} + \big(\theta^2\, E^2 - 4\big)\, \big(\phi_s^\dagger\, \Gamma_s^\dagger\, \phi_{-s}\, \Gamma_s + \phi_{-s}\, \Gamma_s^\dagger\, \phi_s^\dagger\, \Gamma_s\big) \\ & + s\, \big((2-\theta\, E)^2 + 8\sigma\, \theta\, E\big)\, \phi_s^\dagger\, \mathcal{E}\, \phi_{-s} + s\, \big((2+\theta\, E)^2 - 8\sigma\, \theta\, E\big)\, \phi_{-s}\, \mathcal{E}\, \phi_s^\dagger \Big) \,, \end{aligned} \tag{5.7}$$

where we have introduced the matrices

$$(\Gamma_s)_{n,m} = \mathrm{i}\, s\, \sqrt{m+1}\; \delta_{n,m+1} \qquad \text{and} \qquad (\mathcal{E})_{n,m} = \mathrm{i}\, \big(n + \frac12\big)\, \delta_{n,m} \,. \tag{5.8}$$

This action has a similar structure to that of the Euclidean case [30].

The domain of this matrix model is the space of (infinite) matrices $\mathcal{M}$ described in section 4.2. Let us now set $\alpha = 1$, $\beta = 0$ in the interaction term (5.3), $\sigma = 1$ in the free part (5.7), and consider the matrix model at the self-dual point $\theta = +2/E$. The full action is then given by

$$S_\vee = \frac12 \sum_{s=\pm} \mathrm{Tr}\, \Big(4s\, \theta^{-1}\, \phi_s^\dagger\, \mathcal{E}\, \phi_{-s} + \mu^2\, \phi_s^\dagger\, \phi_{-s} + g^2\, \big(\phi_s^\dagger\, \phi_{-s}\big)^2 \Big) \,. \tag{5.9}$$

The matrix model at the other self-dual point $\theta = -2/E$ is gotten by interchanging $\phi_s^\dagger \leftrightarrow \phi_{-s}$ in (5.9). The action (5.9) admits a continuous $GL(\infty) \times GL(\infty)$ symmetry group defined by the transformations

$$\phi_s \;\longmapsto\; \phi_s\, U_s \qquad \text{and} \qquad \phi_s^\dagger \;\longmapsto\; U_{-s}^{-1}\, \phi_s^\dagger \,, \tag{5.10}$$

with $U_\pm \in \mathcal{M} \cap GL(\infty)$. Thus the self-dual matrix model describes an integrable quantum field theory, just as in the Euclidean case [29, 30]. By [28] and the discussion of section 4.1, the unitary $\mathrm{U}(\infty) \times \mathrm{U}(\infty)$ subgroup of this symmetry group consists of matrices $U_\pm$ corresponding to canonical transformations of $\mathbb{R}^2$ along the forward/backward light-cone direction. Note that the self-dual matrix model is also invariant under a discrete $\mathbb{Z}_2$ symmetry group generated by the combined time-reversal and charge conjugation transformation

$$\boldsymbol{CT} \,:\, \big(\phi_s\,,\, \phi_s^\dagger\big) \;\longmapsto\; \big(\phi_{-s}\,,\, \phi_{-s}^\dagger\big) \quad , \quad \theta \;\longmapsto\; -\theta \,. \tag{5.11}$$

6. Generalization to higher dimensions

There is a natural UV/IR-duality invariant extension of our $1+1$-dimensional model to higher dimensional Minkowski spacetime, which combines our result with that of [26] for the Euclidean case. We will demonstrate this in $D = 2d+2$ dimensional Minkowski spacetime with coordinates $\boldsymbol{x} = (x^\mu)$, $\mu = 0,1,\dots,2d+1$, $x^0 = t$ and derivatives $\partial_\mu = \partial/\partial x^\mu$. The extended field theory in D-dimensional Minkowski spacetime has a similar form as before.

The interactions are formally the same as in (2.9), while the free part of the action now reads

$$S_0 = \int_{\mathbb{R}^D} \mathrm{d}\boldsymbol{x}\ \phi^*(\boldsymbol{x})\big(\sigma\,\mathsf{K}^2 + (1-\sigma)\,\tilde{\mathsf{K}}^2 + \mu^2\big)\phi(\boldsymbol{x}) \tag{6.1}$$

with $\mathsf{K}^2 := \frac{1}{2}\,(-\,\mathrm{i}\,\partial_\mu + F_{\mu\nu}\,x^\nu)^2$ and the $D\times D$ antisymmetric electromagnetic tensor in Jordan normal form

$$(F_{\mu\nu}) = \begin{pmatrix} 0 & E & & & & & \\ -E & 0 & & & & & \\ & & 0 & B_1 & & & \\ & & -B_1 & 0 & & & \\ & & & & \ddots & & \\ & & & & & 0 & B_d \\ & & & & & -B_d & 0 \end{pmatrix} \tag{6.2}$$

for $E, B_k > 0$. The differential operator $\tilde{\mathsf{K}}^2$ is defined below. The coordinate system on $\mathbb{R}^D$ is chosen in such a way that the noncommutativity parameter matrix $(\theta^{\mu\nu})$ appearing in the star product is in its canonical skew-diagonal form

$$\big(\theta^{\mu\nu}\big) = \begin{pmatrix} 0 & \theta_0 & & & & & \\ -\theta_0 & 0 & & & & & \\ & & 0 & \theta_1 & & & \\ & & -\theta_1 & 0 & & & \\ & & & & \ddots & & \\ & & & & & 0 & \theta_d \\ & & & & & -\theta_d & 0 \end{pmatrix}\ , \tag{6.3}$$

with $\theta_a > 0$.

With these definitions the operator K^2 decomposes into a sum

$$\mathsf{K}^2 = \mathsf{D}^2 + \sum_{k=1}^{d} \mathsf{D}^2_{\mathrm{E},k} \tag{6.4}$$

of d copies of the Landau Hamiltonian

$$\mathsf{D}^2_{\mathrm{E},k} = \frac{1}{2}\,\Big[-\big(\partial_{2k}^2 + \partial_{2k+1}^2\big) - 2\,\mathrm{i}\,B_k\big(x^{2k+1}\partial_{2k} - x^{2k}\partial_{2k+1}\big) + B_k^2\,\Big(\big(x^{2k}\big)^2 + \big(x^{2k+1}\big)^2\Big)\Big] \tag{6.5}$$

for $k = 1,\dots,d$, and the Klein-Gordon operator D^2 introduced in section 2. The classical duality is now proven in the same way as before. The self-dual point is given by $(\theta^{\mu\nu}) = \pm\,2(F_{\mu\nu})^{-1}$, or equivalently by $\theta_0 = \pm\,2/E$ and $\theta_k = \pm\,2/B_k$, $k = 1,\dots,d$, where the sign has to be the same for all θ_a, $a = 0,1,\dots,d$.

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Eigenfunctions of the operator K^2 are now given by tensor products of eigenfunctions of D^2 and $\mathsf{D}^2_{\mathrm{E},k}$ for $k = 1, \dots, d$, analogously to the Euclidean analysis of [26]. The Landau Hamiltonian $\mathsf{D}^2_{\mathrm{E},k}$ describes the motion of a charged particle in the two-dimensional Euclidean (x^{2k}, x^{2k+1})-plane in the presence of a background magnetic field with field strength $2B_k$, and its eigenfunctions are the well-known Landau wavefunctions $f_{m_k,n_k}(x^{2k}, x^{2k+1})$, $m_k, n_k \in \mathbb{N}_0$. These functions are simultaneous eigenfunctions of the operators $\mathsf{D}^2_{\mathrm{E},k}$ and $\tilde{\mathsf{D}}^2_{\mathrm{E},k} := \mathsf{D}^2_{\mathrm{E},k}\big|_{B_k \to -B_k}$ with eigenvalues $2B_k\,(m_k + \frac12)$ and $2B_k\,(n_k + \frac12)$, respectively. These definitions give rise to a new operator $\tilde{\mathsf{K}}^2$ obtained from K^2 by substituting $\tilde{\mathsf{D}}^2$ for D^2 and $\tilde{\mathsf{D}}^2_{\mathrm{E},k}$ for $\mathsf{D}^2_{\mathrm{E},k}$. Simultaneous generalized eigenfunctions of K^2 and $\tilde{\mathsf{K}}^2$ are therefore given by tensor products

$$f^\pm_{\boldsymbol{p}}(\boldsymbol{x}) = \big(f^\pm_{m_0,n_0} \otimes f_{m_1,n_1} \otimes \cdots \otimes f_{m_d,n_d}\big)(\boldsymbol{x}) \tag{6.6}$$

with $\boldsymbol{p} := (\boldsymbol{m}, \boldsymbol{n}) = (m_0, m_1, \dots, m_d, n_0, n_1, \dots, n_d) \in \mathbb{N}_0^D$. The corresponding generalized eigenvalue equations are

$$\begin{aligned} \mathsf{K}^2 f^\pm_{\boldsymbol{p}} &= \Big(\pm\, 2\,\mathrm{i}\, E\, \Big(m_0 + \frac12\Big) + \sum_{k=1}^d 2B_k\, \Big(m_k + \frac12\Big)\Big)\, f^\pm_{\boldsymbol{p}} \ =: \ E^\pm(\boldsymbol{m})\, f^\pm_{\boldsymbol{p}} \,, \\ \tilde{\mathsf{K}}^2 f^\pm_{\boldsymbol{p}} &= \Big(\pm\, 2\,\mathrm{i}\, E\, \Big(n_0 + \frac12\Big) + \sum_{k=1}^d 2B_k\, \Big(n_k + \frac12\Big)\Big)\, f^\pm_{\boldsymbol{p}} \ =: \ E^\pm(\boldsymbol{n})\, f^\pm_{\boldsymbol{p}} \,. \end{aligned} \tag{6.7}$$

This extended field theory now comprises all the features of our 1+1-dimensional model and the $2d$-dimensional Euclidean model investigated in [26]. Thus it is duality covariant and has a matrix model representation in terms of the *extended Landau basis* defined in (6.6).

The Landau wavefunctions $f_{m,n}(x, y)$ form a basis for $L^2(\mathbb{R}^2)$, which simply reflects the fact that they are the Wigner distributions of the harmonic oscillator eigenoperators $|m\rangle\langle n|$. The extended Landau wavefunctions $f^\pm_{\boldsymbol{p}}$ are Wigner distributions of the tensor products

$$|f^\pm_{m_0}, m_1, \dots, m_d\rangle\langle f^\mp_{n_0}, n_1, \dots, n_d| = |f^\pm_{m_0}\rangle\langle f^\mp_{n_0}| \otimes |m_1\rangle\langle n_1| \otimes \cdots \otimes |m_d\rangle\langle n_d| \,. \tag{6.8}$$

Most of the analysis of the 1+1-dimensional case is now easily generalized to higher dimensions. Each field ϕ in a suitable domain $\Phi \subset \mathcal{S}(\mathbb{R}^D)$, dense in $L^2(\mathbb{R}^D)$, can be decomposed as

$$\phi(\boldsymbol{x}) = \frac12 \sum_{s=\pm}\ \sum_{\boldsymbol{p}\in\mathbb{N}_0^D} f^s_{\boldsymbol{p}}(\boldsymbol{x})\, \phi^{-s}_{\boldsymbol{p}} \qquad \text{with} \quad \phi^s_{\boldsymbol{p}} = \int_{\mathbb{R}^D} \mathrm{d}\boldsymbol{x}\ f^s_{\boldsymbol{p}}(\boldsymbol{x})^*\, \phi(\boldsymbol{x}) \,. \tag{6.9}$$

The free action takes the form

$$S_0 = \frac12 \sum_{s=\pm}\ \sum_{\boldsymbol{p}\in\mathbb{N}_0^D} \big(\sigma\, E^s(\boldsymbol{m}) + (1-\sigma)\, E^s(\boldsymbol{n}) + \mu^2\big)\, \phi^{s\,*}_{\boldsymbol{p}}\, \phi^{-s}_{\boldsymbol{p}} \,, \tag{6.10}$$

and the two propagators in this basis, given by

$$\begin{aligned} C^\pm(\boldsymbol{p}) &= \langle f^\mp_{\boldsymbol{p}}| 2\,\mathrm{i}\, \big(\sigma\, \mathsf{K}^2 + (1-\sigma)\, \tilde{\mathsf{K}}^2 + \mu^2\big)^{-1} |f^\pm_{\boldsymbol{p}}\rangle \\ &= 2\,\mathrm{i}\, \big(\sigma\, E^\pm(\boldsymbol{m}) + (1-\sigma)\, E^\pm(\boldsymbol{n}) + \mu^2\big)^{-1} \,, \end{aligned} \tag{6.11}$$

can be regularized in the same way as before, proving the duality invariance of the higher-dimensional regularized quantum field theory.

Using the noncommutativity parameters $(\theta^{\mu\nu})$ for the Wigner transformations and the star product, the resulting extended Landau wavefunctions obey the same nice projector property under the star product given by

$$f^{\pm}_{(\boldsymbol{m},\boldsymbol{n})}\star f^{\pm}_{(\boldsymbol{m}',\boldsymbol{n}')}=\delta_{\boldsymbol{n},\boldsymbol{m}'}\,f^{\pm}_{(\boldsymbol{m},\boldsymbol{n}')}\,, \tag{6.12}$$

where $\delta_{\boldsymbol{m},\boldsymbol{n}}:=\delta_{m_0,n_0}\,\delta_{m_1,n_1}\cdots\delta_{m_d,n_d}$. As was shown in [30], the Landau Hamiltonians $\mathsf{D}^2_{\mathrm{E},k}$ and $\tilde{\mathsf{D}}^2_{\mathrm{E},k}$ can be written in terms of standard harmonic oscillator creation and annihilation operators for each $k=1,\dots,d$. With the definitions

$$\begin{aligned}
(\mathcal{E}^s_0)_{\boldsymbol{m},\boldsymbol{n}} &= \mathrm{i}\,s\,\left(n_0+\frac{1}{2}\right)\,\delta_{\boldsymbol{m},\boldsymbol{n}}\,,\\
(\mathcal{E}^s_k)_{\boldsymbol{m},\boldsymbol{n}} &= \left(n_k+\frac{1}{2}\right)\,\delta_{\boldsymbol{m},\boldsymbol{n}}\,,\\
(\Gamma^s_0)_{\boldsymbol{m},\boldsymbol{n}} &= \mathrm{i}\,s\,\sqrt{n_0+1}\;\delta_{m_0,n_0+1}\,\delta_{m_1,n_1}\cdots\delta_{m_d,n_d}\,,\\
(\Gamma^s_k)_{\boldsymbol{m},\boldsymbol{n}} &= \sqrt{n_k+1}\;\delta_{m_0,n_0}\cdots\delta_{m_k,n_k+1}\cdots\delta_{m_d,n_d}
\end{aligned} \tag{6.13}$$

along with $F_0=E$ and $F_k=B_k$ for $k=1,\dots,d$, we can thereby map the free part of the D-dimensional $\phi^{\star 4}$ field theory in Minkowski spacetime onto a two-matrix model with action

$$\begin{aligned}
S_0=\frac{1}{8}\sum_{s=\pm}\sum_{a=0}^{d}\frac{1}{\theta_a}\,\mathrm{Tr}\,\Bigg(&\frac{4\theta_a\mu^2}{d+1}\phi_s^\dagger\phi_{-s}+\big(\theta_a^2F_a^2-4\big)\big(\phi_s^\dagger\Gamma^{s\,\dagger}_a\phi_{-s}\Gamma^s_a+\phi_{-s}\Gamma^{s\,\dagger}_a\phi_s^\dagger\Gamma^s_a\big)\\
&+\big((2-\theta_aF_a)^2+8\sigma\theta_aF_a\big)\phi_s^\dagger\mathcal{E}^s_a\phi_{-s}+\big((2+\theta_aF_a)^2-8\sigma\theta_aF_a\big)\phi_{-s}\mathcal{E}^s_a\phi_s^\dagger\Bigg),
\end{aligned} \tag{6.14}$$

where $(\phi_s)_{\boldsymbol{m},\boldsymbol{n}}:=\phi^s_{(\boldsymbol{m},\boldsymbol{n})}$ (regarded as a matrix via lexicographic ordering $\mathbb{N}_0^D\sim\mathbb{N}_0$, for example) and $\mathrm{Tr}\,(\phi_s):=\sum_{\boldsymbol{n}\in\mathbb{N}_0^D}(\phi_s)_{\boldsymbol{n},\boldsymbol{n}}$. Due to (6.12), the interaction terms take the same form as in (5.3). At the self-dual points given by $\theta_a=\pm\,2/F_a$, and with the definition

$$(\mathcal{E}^s)_{\boldsymbol{m},\boldsymbol{n}}=\sum_{a=0}^{d}2F_a\,(\mathcal{E}^s_a)_{\boldsymbol{m},\boldsymbol{n}}=2\Bigg(\mathrm{i}\,s\,E\,\left(n_0+\frac{1}{2}\right)+\sum_{k=1}^{d}B_k\,\left(n_k+\frac{1}{2}\right)\Bigg)\,\delta_{\boldsymbol{m},\boldsymbol{n}}\,, \tag{6.15}$$

we obtain the same self-dual two-matrix model as in section 5.

7. Summary and discussion

In this paper we have proven that a noncommutative $\phi^{\star 4}$-theory, describing a charged scalar boson moving in Minkowski spacetime in the presence of a background electromagnetic field, is invariant under a special UV/IR duality generated by symplectic Fourier transformation of fields. This was achieved by extending the methods which were used in the Euclidean situation [26]. We were able to map our noncommutative field theory onto a two-matrix model and regularize it in a duality invariant fashion. What makes the Minkowskian story much more intricate than the Euclidean one is that the Lorentzian kinetic operator has a continuous spectrum extended over the whole real line. By analytically continuing its eigenfunctions into the complex energy plane and closing the integration contour of the continuous eigenfunction expansion on an infinite arc in the upper or lower complex half-plane, we get two distinct discrete expansions from the isolated poles on the imaginary axis. A determination of the resulting generalized functions shows that they are given by Wick rotated Landau wavefunctions, including a Wick rotation of the background field. One expansion corresponds to the Wick rotation $(x, E) \to (\mathrm{i}\,t, \mathrm{i}\,E)$ and the other expansion to $(x, E) \to (-\mathrm{i}\,t, -\mathrm{i}\,E)$. This shows that we can map one expansion to the other by a combined time-reversal plus charge conjugation transformation $\boldsymbol{CT}$. This suggests that the corresponding propagators relate the propagation of charged particles and antiparticles in different time directions respectively. We found an explicit expression for both propagators in a background electric field alone, and determined explicitly the appropriate domain for the expansion of noncommutative fields in these "electric Landau wavefunctions" in terms of Gel'fand-Shilov spaces.

However, a non-trivial result, which doesn't simply follow by Wick rotation, is that stability of the theory requires the use of both expansions simultaneously, i.e. we have to make the expansion in a $\boldsymbol{CT}$-invariant way. This shows that in Minkowski spacetime we effectively require twice as many degrees of freedom as compared to the Euclidean case. This is most apparent in the matrix model representation. While the Euclidean field theory is a one-matrix model, the Lorentzian field theory is a two-matrix model.

This new matrix basis could now be used to implement the renormalization programme for noncommutative field theory in Minkowski spacetime. In the same way as the Landau basis was a crucial ingredient in the proof of the renormalizability of some Euclidean noncommutative field theories, the electric Landau basis could be used in similar theories formulated in Minkowski spacetime. One could first examine the Minkowskian version of original Grosse-Wulkenhaar model [22], which consists in adding an inverted harmonic oscillator potential to the kinetic term of a real $\phi^{\star 4}$-theory, as given by the operator (2.15). The corresponding propagator requires inversion of the analog of the matrix appearing in (5.7) at $\sigma = \frac{1}{2}$, and can be found using the techniques of [17]. In this way our formalism describes the appropriate analytic continuation of the Grosse-Wulkenhaar models to Minkowski signature. Along these lines it is interesting to explore the structure of the presumably inequivalent quantization of the duality covariant field theory using the S-matrix formalism in our two-matrix basis. It would also be interesting to see if the exactly solvable self-dual matrix models of section 5 lead to any different nonperturbative renormalizability properties compared to the Euclidean case [29]. All of these interesting renormalization issues are left for future investigations.

We conclude by pointing out an interesting but somewhat unrelated offspring of our analysis. A corollary of our work is a rigorous mathematical proof of the electric-magnetic duality of the QED effective action, which states that one can obtain the effective action of charged particles in an electric background E from that of charged particles in a magnetic

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background B by the substitution $B \to \mathrm{i}\,E$ [11]. The effective action is simply given by

$$\begin{aligned} S_{\text{eff}} &= \mathrm{i}\,\log\left(\int_\Phi \mathcal{D}\phi\ \mathcal{D}\phi^*\ \mathrm{e}^{\,\mathrm{i}\,S_0|_{\sigma=1}}\right) \\ &= -\,\mathrm{i}\,\log\det\left(\mathsf{D}^2+\mu^2\right) \\ &= -\sum_{s=\pm}\ \sum_{n=0}^\infty \mathrm{i}\,\log\left(2\,\mathrm{i}\,s\,E\left(n+\frac12\right)+\mu^2\right)\ , \end{aligned} \tag{7.1}$$

where we have omitted the infinite vacuum contribution in the second line and used the generalized discrete spectrum in the third line. The techniques developed in this paper may have further applications in this context.

Acknowledgments

We thank D. Bahns, L. Boulton, H. Grosse, E. Langmann, H. Steinacker and K. Vogeler for helpful discussions. This work was supported in part by the EU-RTN Network Grant MRTN-CT-2004-005104.

A. Generalized eigenfunctions

Given the relation with the inverted harmonic oscillator, it is natural to introduce analogs of the standard ladder operators. Defining

$$\hat{\boldsymbol{a}}^\pm = \frac{1}{\sqrt{2\theta}}\left(\hat{\boldsymbol{p}}\mp\hat{\boldsymbol{q}}\right)\ , \tag{A.1}$$

one has the commutation relations

$$\left[\hat{\boldsymbol{a}}^-\ ,\ \hat{\boldsymbol{a}}^+\right] = \mathrm{i} \tag{A.2}$$

and the operator $\hat{\boldsymbol{H}}$ can be represented as

$$\hat{\boldsymbol{H}} = \frac{\theta}{2}\left(\hat{\boldsymbol{a}}^+\,\hat{\boldsymbol{a}}^- + \hat{\boldsymbol{a}}^-\,\hat{\boldsymbol{a}}^+\right)\ . \tag{A.3}$$

These operators are not ladder operators in the usual sense, since they are not Hermitean conjugates of one another. Nevertheless, we can construct our basis distributions $|f_n^\pm\rangle$ and $\langle f_n^\pm|$ in vacuum representations of the algebra (A.2) defined by applying these operators to states $|0,\pm\rangle$ and $\langle 0,\pm|$, respectively, which are determined via the conditions

$$\begin{aligned} \hat{\boldsymbol{a}}^{-s}|0,s\rangle &= 0\,, \\ \langle 0,s|\hat{\boldsymbol{a}}^{-s} &= 0\,, \\ \langle 0,s|0,-s\rangle &= 1 \end{aligned} \tag{A.4}$$

for $s=\pm$. The inner product in (A.4) follows again from the fact that $\langle f_n^\pm| = \langle n|\hat{\boldsymbol{V}}_\mp^{-1}$ is only orthonormal to $|f_n^\mp\rangle$. The generalized eigenstates

$$\begin{aligned} |n,+\rangle &:= \frac{(-\,\mathrm{i}\,)^n}{\sqrt{n!}}\left(\hat{\boldsymbol{a}}^+\right)^n|0,+\rangle\ , \\ |n,-\rangle &:= \frac{1}{\sqrt{n!}}\left(\hat{\boldsymbol{a}}^-\right)^n|0,-\rangle\,, \end{aligned}$$

$$
\begin{aligned}
\langle n,+| &:= \frac{\mathrm{i}^{\,n}}{\sqrt{n!}}\,\langle 0,+|\big(\hat{\boldsymbol{a}}^{+}\big)^{n}\,, \\
\langle n,-| &:= \frac{1}{\sqrt{n!}}\,\langle 0,-|\big(\hat{\boldsymbol{a}}^{-}\big)^{n}
\end{aligned} \tag{A.5}
$$

have the desired properties

$$
\begin{aligned}
\hat{\boldsymbol{H}}|n,\pm\rangle &= \pm\,\mathrm{i}\,\theta\,\big(n+\tfrac{1}{2}\big)\,|n,\pm\rangle\,, \\
\langle n,\pm|\hat{\boldsymbol{H}} &= \mp\,\mathrm{i}\,\theta\,\big(n+\tfrac{1}{2}\big)\,\langle n,\pm|\,, \\
\langle n,\pm|m,\mp\rangle &= \delta_{nm}\ .
\end{aligned} \tag{A.6}
$$

Consequently these states coincide (up to a phase factor) with the distributions $|f_n^{\pm}\rangle = \hat{\boldsymbol{V}}_{\pm}|n\rangle$ and $\langle f_n^{\pm}| = \langle n|\hat{\boldsymbol{V}}_{\mp}^{-1}$ constructed using the complex scaling of section 3.2.

The operators $\hat{\boldsymbol{a}}^{\pm}$ together with the Weyl-Wigner correspondence now allow us to construct the generalized functions $f_{n,m}^{\pm}(\boldsymbol{x})$ formally via

$$
f_{n,m}^{\pm} = \mathsf{W}[\,|n,\pm\rangle\langle m,\mp|\,] = \frac{(\mp\,\mathrm{i}\,)^{n}}{\sqrt{n!\,m!}}\ \mathsf{W}[\hat{\boldsymbol{a}}^{\pm}]^{\star n}\star f_{0,0}^{\pm}\star \mathsf{W}[\hat{\boldsymbol{a}}^{\mp}]^{\star m}\,, \tag{A.7}
$$

where $\mathsf{W}[\hat{\boldsymbol{a}}^{s}]^{\star n}$ denotes the n-fold star product $\mathsf{W}[\hat{\boldsymbol{a}}^{s}]\star\cdots\star\mathsf{W}[\hat{\boldsymbol{a}}^{s}]$. With the notation $x^{\pm}=t\pm x$ and $\partial_{\pm}=\partial_t\pm\partial_x$, we find for an arbitrary function $f(\boldsymbol{x})$ the star products

$$
\begin{aligned}
\mathsf{W}[\hat{\boldsymbol{a}}^{\pm}](\boldsymbol{x})\star f(\boldsymbol{x}) &= \frac{\mathrm{i}}{2}\left(-\sqrt{\frac{\theta}{2}}\ \partial_{\pm}\pm\,\mathrm{i}\sqrt{\frac{2}{\theta}}\ x^{\mp}\right)f(\boldsymbol{x})\,, \\
f(\boldsymbol{x})\star\mathsf{W}[\hat{\boldsymbol{a}}^{\mp}](\boldsymbol{x}) &= \frac{\mathrm{i}}{2}\left(\sqrt{\frac{\theta}{2}}\ \partial_{\mp}\mp\,\mathrm{i}\sqrt{\frac{2}{\theta}}\ x^{\pm}\right)f(\boldsymbol{x})\ .
\end{aligned} \tag{A.8}
$$

This motivates the definition of new "ladder operators" on $\boldsymbol{x}$-space given by

$$
a_1^{\pm}=\frac{\mathrm{i}}{2}\left(-\sqrt{\frac{\theta}{2}}\ \partial_{\pm}\pm\,\mathrm{i}\sqrt{\frac{2}{\theta}}\ x^{\mp}\right)\qquad\text{and}\qquad a_2^{\pm}=\frac{\mathrm{i}}{2}\left(\sqrt{\frac{\theta}{2}}\ \partial_{\mp}\mp\,\mathrm{i}\sqrt{\frac{2}{\theta}}\ x^{\pm}\right)\ . \tag{A.9}
$$

The new operators $a_i^{\pm}$, $i=1,2$ obey the nonvanishing commutation relations $[a_i^{-},a_j^{+}] = \mathrm{i}\,\delta_{ij}$, and with $\theta=2/E$ our basic differential operators can be expressed as

$$
\mathsf{D}^2 = 2E\left(a_1^{+}\,a_1^{-}+\frac{\mathrm{i}}{2}\right)\qquad\text{and}\qquad \tilde{\mathsf{D}}^2 = 2E\left(a_2^{+}\,a_2^{-}+\frac{\mathrm{i}}{2}\right). \tag{A.10}
$$

The conditions (A.4) translated into this language respectively give the differential equations

$$
a_1^{-}\,f_{0,0}^{+}(\boldsymbol{x}) = a_2^{-}\,f_{0,0}^{+}(\boldsymbol{x}) = 0\qquad\text{and}\qquad a_1^{+}\,f_{0,0}^{-}(\boldsymbol{x}) = a_2^{+}\,f_{0,0}^{-}(\boldsymbol{x}) = 0\,, \tag{A.11}
$$

which can each be solved to give

$$
f_{0,0}^{\pm}(\boldsymbol{x}) = \frac{\mathrm{i}\,E}{\pi}\ \mathrm{e}^{\,\mp\,\mathrm{i}\,E\,(t^2-x^2)} \tag{A.12}
$$

in the space $\mathcal{S}'(\mathbb{R}^2)$, where the normalization constant has been fixed by $\int_{\mathbb{R}^2}\ \mathrm{d}\boldsymbol{x}\ f_{0,0}^{\pm}(\boldsymbol{x}) = 1$.

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Lemma 6. *The generalized functions* $f^{s,s'}_{n,m} = \mathsf{W}[\,|f^s_n\rangle\langle f^{-s'}_m|\,]$ *vanish for distinct* $s, s' = \pm$.

Proof. The analog of (A.11) for $f^{+,-}_{0,0}$ yields the two differential equations

$$\partial_- f^{+,-}_{0,0}(\boldsymbol{x}) = -\,\mathrm{i}\,E\,x^+\,f^{+,-}_{0,0}(\boldsymbol{x}) \qquad \text{and} \qquad \partial_- f^{+,-}_{0,0}(\boldsymbol{x}) = +\,\mathrm{i}\,E\,x^+\,f^{+,-}_{0,0}(\boldsymbol{x})\,, \quad \text{(A.13)}$$

which together imply that $f^{+,-}_{0,0} = 0$ by continuity. The same argument leads to $f^{-,+}_{0,0} = 0$. The result now follows from the analog of (A.7). □

We will now show that the explicit forms of the generalized eigenfunctions $f^{\pm}_{m,n}(\boldsymbol{x})$ in Minkowski signature are simply given by the Landau wavefunctions with Wick rotated parameters.

Proposition 1. *The generalized eigenfunctions can be written as*

$$f^{\pm}_{m,n}(z,\varphi) = \frac{|E|}{2\pi}\sqrt{\frac{n!}{m!}}(-1)^n\,\mathrm{e}^{\mp\,\mathrm{i}\,Ez^2/2}(\pm\,\mathrm{i}\,E)^{(m-n)/2}\,z^{m-n}\,\mathrm{e}^{\mp\varphi(m-n)}\,L^{m-n}_n\left(\pm\,\mathrm{i}\,Ez^2\right) \quad \text{(A.14)}$$

$$= \frac{|E|}{2\pi}\sqrt{\frac{m!}{n!}}(-1)^m\,\mathrm{e}^{\mp\,\mathrm{i}\,Ez^2/2}(\pm\,\mathrm{i}\,E)^{(n-m)/2}\,z^{n-m}\,\mathrm{e}^{\mp\varphi(m-n)}\,L^{n-m}_m\left(\pm\,\mathrm{i}\,Ez^2\right), \text{(A.15)}$$

where $z = \sqrt{t^2 - x^2}$, $\varphi = \tanh^{-1}(x/t)$ *and* $L^k_n(y)$ *are the associated Laguerre polynomials.*

Proof. We use the Wigner transformation formula (3.50) and the explicit form of the generalized eigenfunctions (3.26) with the electric field $E' = E/2 = 1/\theta$. Using the generating function for the Hermite polynomials given by

$$\mathrm{e}^{\frac{-\xi+\xi\,q}{\mp\,\mathrm{i}\,\theta}} = \sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{\xi}{\sqrt{\mp\,\mathrm{i}\,\theta}}\right)^n H_n\big(q/\sqrt{\mp\,\mathrm{i}\,\theta}\,\big)\,, \qquad \text{(A.16)}$$

we have

$$\begin{aligned} K^{\pm}(\xi,\eta;t,x) &:= 2\pi\,|\theta|\sum_{m,n=0}^{\infty}\sqrt{\frac{2^{m+n}}{m!\,n!}}\left(\frac{\xi}{\sqrt{\mp\,\mathrm{i}\,\theta}}\right)^m\left(\frac{\eta}{\sqrt{\mp\,\mathrm{i}\,\theta}}\right)^n f^{\pm}_{m,n}(\boldsymbol{x}) \\ &= \frac{1}{\sqrt{\mp\,\mathrm{i}\,\theta\,\pi}}\int_{\mathbb{R}}\mathrm{d}k\ \exp\Big\{-\frac{1}{\mp\,\mathrm{i}\,\theta}\Big[\big(\xi^2-2\xi\,(t+k/2)\big)-\big(\eta^2-2\eta\,(t-k/2)\big) \\ &\qquad -\frac{1}{2}(t+k/2)^2-\frac{1}{2}(t-k/2)^2-\mathrm{i}\,kx\Big]\Big\}. \end{aligned} \quad \text{(A.17)}$$

Evaluating the formal Gaussian integral in (A.17) gives finally the generating function

$$K^{\pm}(\xi,\eta;t,x) = 2\exp\left\{\frac{1}{\mp\,\mathrm{i}\,\theta}\,\left[x^2-t^2+2\xi\,(t\mp x)+2\eta\,(t\pm x)-2\eta\,\xi\right]\right\} \qquad \text{(A.18)}$$

in the space $\mathcal{S}'(\mathbb{R}^4)$. The generalized functions $f^{\pm}_{m,n}(\boldsymbol{x})$ can now be obtained by taking suitable derivatives of (A.18) with respect to the variables ξ and η.

For $m \geq n$ one finds

$$f^{\pm}_{m,n}(\boldsymbol{x}) = \frac{1}{2\pi\,|\theta|}\,\frac{1}{\sqrt{m!\,n!}}\left(\frac{\mp\,\mathrm{i}\,\theta}{2}\right)^{(m+n)/2}\frac{\partial^m}{\partial\xi^m}\,\frac{\partial^n}{\partial\eta^n}K^{\pm}(\xi,\eta;t,x)\Big|_{\xi=\eta=0}$$

$$= \frac{\sqrt{m!\,n!}}{\pi\,|\theta|}\ \mathrm{e}^{\,\frac{1}{\mp\mathrm{i}\,\theta}\,(x^2-t^2)}\ \left(\frac{2}{\mp\mathrm{i}\,\theta}\right)^{(m-n)/2}\ (t\mp x)^{m-n}$$
$$\times\sum_{p=0}^{n}\left(\frac{2}{\mp\mathrm{i}\,\theta}\left(t^2-x^2\right)\right)^{n-p}\ \frac{(-1)^p}{(m-p)!\,(n-p)!\,p!}\ . \tag{A.19}$$

This expression can be further simplified by introducing the rapidity parameters (z,φ) defined by

$$t = z\,\cosh\varphi \qquad \text{and} \qquad x = z\,\sinh\varphi\,, \tag{A.20}$$

such that $x^\pm = t\pm x = z\ \mathrm{e}^{\pm\varphi}$. Introducing the summation index $q=n-p$, inserting $E=2/\theta$ and using the definition of the associated Laguerre functions

$$L_n^k(y) = \sum_{q=0}^{n}\frac{(n+k)!\,(-1)^q\,y^q}{(n-q)!\,(k+q)!\,q!}\,, \tag{A.21}$$

we arrive at the explicit form for $m\geq n$ given by (A.14). The calculation for $m<n$ is completely analogous and leads to (A.15). However, using the identity [24, p. 321]

$$(-1)^n\,r^{m-n}\,L_n^{m-n}\big(r^2\big) = (-1)^m\,r^{n-m}\,L_m^{n-m}\big(r^2\big)\,, \tag{A.22}$$

we see that both forms (A.14) and (A.15) are valid for *generic* $m,n\in\mathbb{N}_0$, and are thus equivalent. □

B. Free two-point function

In this appendix we will derive an explicit expression for the free propagator (3.56), (3.57) at $\sigma=1$ in the spacetime coordinate basis. We begin with a spectral expansion of the propagator

$$C^\pm_{\sigma=1}(\boldsymbol{x},\boldsymbol{x}'\,) = 2\,\mathrm{i}\,\langle\boldsymbol{x}|\big(\mathsf{D}^2+\mu^2\big)^{-1}|\boldsymbol{x}'\,\rangle = \sum_{m,n=0}^{\infty}\frac{2\,\mathrm{i}\,f^\pm_{m,n}(\boldsymbol{x})\,f^\pm_{n,m}(\boldsymbol{x}'\,)}{\pm\,\mathcal{E}_m+\mu^2}\,, \tag{B.1}$$

where we have used (3.50). First we will evaluate the sum over n. Substituting the expression (A.14) for $f^\pm_{m,n}(\boldsymbol{x})$ and (A.15) for $f^\pm_{n,m}(\boldsymbol{x}'\,)$ we get

$$\sum_{n=0}^{\infty} f^\pm_{m,n}(\boldsymbol{x})f^\pm_{n,m}(\boldsymbol{x}') = \left(\frac{1}{\pi\theta}\right)^2\ \mathrm{e}^{\,\mp\,\mathrm{i}\,E(z^2+z'^{\,2})/2}\frac{(\pm\,\mathrm{i}\,Ezz')^m}{m!}\ \mathrm{e}^{\,\mp(\varphi-\varphi')m} \tag{B.2}$$
$$\times\sum_{n=0}^{\infty} n!\big(\pm\,\mathrm{i}\,Ezz'\ \mathrm{e}^{\,\mp(\varphi-\varphi')}\big)^{-n}L_n^{m-n}\big(\pm\,\mathrm{i}\,Ez^2\big)L_n^{m-n}\big(\pm\,\mathrm{i}\,Ez'^{\,2}\big).$$

Using the identity [24, eq. (48.23.11)]

$$\sum_{n=0}^{\infty} n!\,c^n\,L_n^{m-n}(\xi)\,L_n^{k-n}(\eta) = k!\ \mathrm{e}^{\,c\,\xi\,\eta}\,(1-\eta\,c)^{m-k}\,c^m\,L_k^{m-k}\bigg(\frac{(1-\xi\,c)\,(\eta\,c-1)}{c}\bigg) \tag{B.3}$$

for $k = m$, after a bit of algebra we find

$$\sum_{n=0}^{\infty} f^{\pm}_{m,n}(\boldsymbol{x}) f^{\pm}_{n,m}(\boldsymbol{x}') = \frac{1}{\pi^2\theta^2}\ \mathrm{e}^{\mp\,\mathrm{i}\,E(\boldsymbol{x}-\boldsymbol{x}')^2/2}\ \mathrm{e}^{-\,\mathrm{i}\,zz'\sinh(\varphi'-\varphi)} L_m\big(\pm\,\mathrm{i}\,E(\boldsymbol{x}-\boldsymbol{x}')^2\big), \quad \text{(B.4)}$$

where the factor $m!\,c^m$ coming from (B.3) cancels the same factor appearing in the denominator of (B.2).

To obtain the full propagator we also have to carry out the summation over the index m to get

$$C^{\pm}_{\sigma=1}(\boldsymbol{x},\boldsymbol{x}') = \frac{1}{\pi^2\theta^2}\ \mathrm{e}^{\mp\,\mathrm{i}\,E(\boldsymbol{x}-\boldsymbol{x}')^2/2}\ \mathrm{e}^{-\,\mathrm{i}\,Ezz'\sinh(\varphi'-\varphi)} \sum_{m=0}^{\infty} \frac{L_m\big(\pm\,\mathrm{i}\,E(\boldsymbol{x}-\boldsymbol{x}')^2\big)}{\pm E\big(m+\frac{1}{2}\pm\frac{\mathrm{i}\,\mu^2}{2E}\big)}. \quad \text{(B.5)}$$

Using the identity [24, eq. (48.2.3)]

$$\sum_{m=0}^{\infty} \frac{1}{m+a}\, L^k_m(w) = \frac{\Gamma(a)\,\Gamma(k)}{\Gamma(a-k)}\ {}_1F_1(a;k+1;w) + \Gamma(k)\ {}_1F_1(a-k;1-k;w) \quad \text{(B.6)}$$

with $a = \frac{1}{2} \pm \frac{\mathrm{i}\,\mu^2}{2E}$, $k=0$, $w = \pm\,\mathrm{i}\,E\,(\boldsymbol{x}-\boldsymbol{x}'\,)^2$ and ${}_1F_1(a;b;w)$ a confluent hypergeometric function, we finally obtain

$$\begin{aligned} C^{\pm}_{\sigma=1}(\boldsymbol{x},\boldsymbol{x}'\,) = \pm\,\frac{1}{2\pi^2}\,E\ \mathrm{e}^{\mp\,\mathrm{i}\,E\,(\boldsymbol{x}-\boldsymbol{x}'\,)^2/2}\ \mathrm{e}^{-\,\mathrm{i}\,E\,z\,z'\sinh(\varphi'-\varphi)}\ {}_1 \\ \times F_1\left(\frac{1}{2}\pm\frac{\mathrm{i}\,\mu^2}{2E}\,;\,1\,;\,\pm\,\mathrm{i}\,E\,(\boldsymbol{x}-\boldsymbol{x}'\,)^2\right) \end{aligned} \quad \text{(B.7)}$$

where we have again substituted $\theta = 2/E$. The domain of validity for the identities used above can be found in [24], and applied in our instance as equalities in the space $\mathcal{S}'(\mathbb{R}^4)$. Note that the factor $\mathrm{e}^{-\,\mathrm{i}\,E\,z\,z'\sinh(\varphi'-\varphi)}$ breaks translation invariance, as expected in an electric background.

References

[1] D. Bahns, *Perturbative methods on the noncommutative Minkowski space*, Ph.D. thesis, DESY-THESIS-2004-004 (2004).

[2] D. Bahns, S. Doplicher, K. Fredenhagen and G. Piacitelli, *On the unitarity problem in space/time noncommutative theories*, *Phys. Lett.* **B 533** (2002) 178 [hep-th/0201222].

[3] P. Bieliavsky, R. Gurau and V. Rivasseau, *Non commutative field theory on rank one symmetric spaces*, arXiv:0806.4255.

[4] E. Bruning and S. Nagamachi, *Relativistic quantum field theory with a fundamental length*, *J. Math. Phys.* **45** (2004) 2199.

[5] O. Civitarese and M. Gadella, *Physical and mathematical aspects of Gamow states*, *Phys. Rept.* **396** (2004) 41.

[6] D. Chruściński, *Wigner functions for damped systems*, math-ph/0209008.

[7] D. Chruściński, *Quantum mechanics of damped systems*, *J. Math. Phys.* **44** (2003) 3718 [math-ph/0301024].

[8] D. Chruściński, *Quantum mechanics of damped systems. II. Damping and parabolic potential barrier*, *J. Math. Phys.* **45** (2004) 841 [math-ph/0307047].

[9] M. Chaichian, M. Mnatsakanova, A. Tureanu and Y. Vernov, *Test functions space in noncommutative quantum field theory*, *JHEP* **09** (2008) 125 [arXiv:0706.1712].

[10] M. Disertori, R. Gurau, J. Magnen and V. Rivasseau, *Vanishing of β-function of non commutative ϕ_4^4 theory to all orders*, *Phys. Lett.* **B 649** (2007) 95 [hep-th/0612251].

[11] G.V. Dunne and T. Hall. *On the QED effective action in time-dependent electric backgrounds*, *Phys. Rev.* **D 58** (1998) 105022 [hep-th/9807031].

[12] R. de la Madrid, *The role of the rigged Hilbert space in quantum mechanics*, *Eur. J. Phys.* **26** (2005) 287 [quant-ph/0502053].

[13] J.B. Geloun, R. Gurau and V. Rivasseau, *Vanishing β-function for Grosse-Wulkenhaar model in a magnetic field*, *Phys. Lett.* **B 671** (2009) 284 [arXiv:0805.4362].

[14] R. Gurau, J. Magnen, V. Rivasseau and F. Vignes-Tourneret, *Renormalization of non-commutative ϕ_4^4 field theory in x space*, *Commun. Math. Phys.* **267** (2006) 515 [hep-th/0512271].

[15] J.B. Geloun and V. Rivasseau, *Color Grosse-Wulkenhaar models: one-loop β- functions*, *Eur. Phys. J.* **C 58** (2008) 115 [arXiv:0805.2538].

[16] H.J. Groenewold, *On the principles of elementary quantum mechanics*, *Physica* **12** (1946) 405.

[17] R. Gurau, V. Rivasseau and F. Vignes-Tourneret, *Propagators for noncommutative field theories*, *Ann. Henri Poincaré* **7** (2006) 1601 [hep-th/0512071].

[18] H. Grosse and H. Steinacker, *A nontrivial solvable noncommutative ϕ^3 model in* 4 *dimensions*, *JHEP* **08** (2006) 008 [hep-th/0603052].

[19] H. Grosse and H. Steinacker, *Renormalization of the noncommutative ϕ^3 model through the Kontsevich model*, *Nucl. Phys.* **B 746** (2006) 202 [hep-th/0512203].

[20] H. Grosse, H. Steinacker and M. Wohlgenannt, *Emergent gravity, matrix models and UV/IR mixing*, *JHEP* **04** (2008) 023 [arXiv:0802.0973].

[21] I.M. Gel'fand and N.Y. Vilenkin, *Generalized functions*, volume 4, Academic Press, (1964).

[22] H. Grosse and R. Wulkenhaar, *Renormalisation of ϕ^4 theory on noncommutative R^2 in the matrix base*, *JHEP* **12** (2003) 019 [hep-th/0307017].

[23] H. Grosse and R. Wulkenhaar, *Renormalisation of ϕ^4 theory on noncommutative R^4 in the matrix base*, *Commun. Math. Phys.* **256** (2005) 305 [hep-th/0401128].

[24] E. Hanson, *A table of series and products*, Prentice-Hall Inc., Englewood Cliffs U.S.A. (1975).

[25] Z. Lozanov-Crvenković and D. Perišić, *Hermite expansions of elements of Gel'fand-Shilov spaces in quasianalytic and non-quasianalytic case*, *Novi Sad J. Math.* **37** (2007) 129 [quant-ph/0701041].

[26] E. Langmann and R.J. Szabo, *Duality in scalar field theory on noncommutative phase spaces*, *Phys. Lett.* **B 533** (2002) 168 [hep-th/0202039].

[27] Y. Liao and K. Sibold, *Time-ordered perturbation theory on noncommutative spacetime: basic rules*, *Eur. Phys. J.* **C 25** (2002) 469 [hep-th/0205269].

JHEP02(2009)031

[28] F. Lizzi, R.J. Szabo and A. Zampini, *Geometry of the gauge algebra in noncommutative Yang-Mills theory*, *JHEP* **08** (2001) 032 [hep-th/0107115].

[29] E. Langmann, R.J. Szabo and K. Zarembo, *Exact solution of noncommutative field theory in background magnetic fields*, *Phys. Lett.* **B 569** (2003) 95 [hep-th/0303082].

[30] E. Langmann, R.J. Szabo and K. Zarembo, *Exact solution of quantum field theory on noncommutative phase spaces*, *JHEP* **01** (2004) 017 [hep-th/0308043].

[31] A. Lakhoua, F. Vignes-Tourneret and J.-C. Wallet, *One-loop β-functions for the orientable non-commutative Gross-Neveu model*, *Eur. Phys. J.* **C 52** (2007) 735 [hep-th/0701170].

[32] S. Minwalla, M. Van Raamsdonk and N. Seiberg, *Noncommutative perturbative dynamics*, *JHEP* **02** (2000) 020 [hep-th/9912072].

[33] S.-J. Rey, *Exact answers to approximate questions: noncommutative dipoles, open Wilson lines, and UV/IR duality*, published in *Les Houches* 2001, *Gravity, gauge theories and strings*, (2002), pg. 587 [hep-th/0207108].

[34] V. Rivasseau, *Noncommutative renormalization*, *Sém. Poincaré* **X** (2007) 15 [arXiv:0705.0705].

[35] V. Rivasseau, *Why renormalizable noncommutative quantum field theories?*, arXiv:0711.1748.

[36] C.-H. Rim and J.H. Yee, *Unitarity in space-time noncommutative field theories*, *Phys. Lett.* **B 574** (2003) 111 [hep-th/0205193].

[37] M.A. Soloviev, *Noncommutativity and θ-locality*, *J. Phys.* **A 40** (2007) 14593 [arXiv:0708.1151].

[38] M.A. Soloviev, *Star product algebras of test functions*, *Theor. Math. Phys.* **153** (2007) 1351 [*Teor. Mat. Fiz.* **153** (2007) 3] [arXiv:0708.0811].

[39] H. Steinacker, *Emergent gravity from noncommutative gauge theory*, *JHEP* **12** (2007) 049 [arXiv:0708.2426].

[40] R.J. Szabo, *Quantum field theory on noncommutative spaces*, *Phys. Rept.* **378** (2003) 207 [hep-th/0109162].

[41] N. Teofanov, *Modulation spaces, Gel'fand-Shilov spaces and pseudodifferential operators*, *Sampl. Theory Signal Image Process.* **5** (2006) 225.

[42] M. Van Raamsdonk, *The meaning of infrared singularities in noncommutative gauge theories*, *JHEP* **11** (2001) 006 [hep-th/0110093].

[43] F. Vignes-Tourneret, *Renormalization of the orientable noncommutative Gross-Neveu model*, *Ann. Henri Poincaré* **8** (2007) 427 [math-ph/0606069].

[44] Z. Wang and S. Wan, *Renormalization of orientable noncommutative complex ϕ^6_3 model*, *Ann. Henri Poincaré* **9** (2008) 65.

JHEP02(2009)031